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# A passivity based synchronization between two different chaotic systems

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**In this paper, we propose a new passivity-based synchronization method for two different chaotic systems. Based on Lyapunov stability theory and linear matrix inequality (LMI) approach, the passivity-based controller is presented to make the synchronization error system between two different chaotic systems not only passive but also asymptotically stable. It is shown that the proposed controller can be obtained by solving the LMI, which can be easily facilitated by using some standard numerical packages. As an application of the proposed method, the synchronization problem between Rossler system and Genesio-Tesi system is investigated.**

**Key words:** Passivity-based synchronization, two different chaotic systems, linear matrix inequality (LMI), Lyapunov stability theory.

## INTRODUCTION

During the last two decades, synchronization in chaotic dynamic systems has received a great deal of interest among scientists from various research fields since Pecora and Carroll (Pecora and Carroll, 1990) introduced a method to synchronize two identical chaotic systems with different initial conditions. It has been widely explored in a variety of fields including physical, chemical and ecological systems (Chen and Dong, 1998). In the literature, various synchronization schemes, such as variable structure control (Wang and Su, 2004), OGY method (Ott et al., 1990), parameters adaptive control (Park, 2005 ; Wang et al., 2003), observer-based control (Yang and Chen, 2002), fuzzy logic approach (Ahn, 2010a, 2010c), backstepping design technique (Hu et al., 2005),  $H_\infty$  approach (Ahn, 2009, 2010b) and so on, have been successfully applied to the chaos synchronization.

The concept of passivity for nonlinear systems attracted new interest in nonlinear system control. The passivity theory plays an important role in designing asymptotically stabilizing controller for nonlinear systems. Wen (1999) applied the passivity technique to design the controller, whose structure is of linear feedback form, to control the Lorenz system. Passivitybased controls for chaotic L<sup>u</sup> system and chaotic oscillations in power system were pro-

posed in (Kemih et al., 2006) and (Wei and Luo, 2007), respectively. Wang and Liu (2007) also applied this technique to design a controller to control a unified chaotic system to zero and any desired equilibrium. Recently, passivity-based controls for hyperchaotic Lorenz system, hyperchaotic Chen system and nuclear spin generator chaotic system were proposed in (Wang and Liu, 2006; Jiao and An, 2008; Kemih, 2009), respectively.

Most synchronization methods are focused on synchronizing two identical chaotic systems. However, experimental and even more real systems are often not fully identical. In many practical systems such as laser array, biological systems and cognitive processes, it is hardly the case that every component can be assumed to be identical. It is thus important and also interesting to investigate synchronization behavior between two different chaotic systems. In this regard, some control methods, such as active control (Yassen, 2005b), backstepping control (Li et al., 2006), dynamic feedback control (Park, 2009), converse Lyapunov approach (Chen et al., 2009) and adaptive control (Li et al., 2007; Salarieh and Shahrokhi, 2008), were proposed for synchronizing two different chaotic systems. To the best of our knowledge, however, for the passivity-based synchronization between two different chaotic systems, there is no result in the literature so far, which still remains challenging.

In this paper, a new controller for the passivity-based

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synchronization of two different chaotic systems is proposed. Theoretical proof revealed that the use of the proposed controller could make the synchronization error system passive and also asymptotically stable. In contrast to other existing results (Yassen, 2005b; Li et al., 2006; Park, 2009; Chen et al., 2009; Yassen, 2005a; Zhang et al., 2006; Li et al., 2007; Salarieh and Shahrokh, 2008), an advantage of the proposed method is the design flexibility it offers. Any smooth function satisfying some condition can stabilize the synchronization error system. Based on Lyapunov method and linear matrix inequality (LMI) approach, an existence criterion for the proposed controller is represented in terms of an LMI. The LMI problem can be solved efficiently by using recently developed convex optimization algorithms (Boyd et al., 1994).

This paper is organized as follows. In Section 2, the basic concept of passivity is introduced. In Section 3, we formulate the problem. In Section 4, an LMI problem for the passivity-based synchronization of two different chaotic systems is proposed. In Section 5, a numerical example is given and finally, conclusions are presented in Section 6.

## BASIC CONCEPT OF PASSIVITY

Consider the following differential equation:

$$\dot{X}(t) = F(X(t)) + G(X(t))U(t), \quad (1)$$

$$Y(t) = H(X(t)), \quad (2)$$

where  $X(t) \in \mathfrak{R}^n$  is the state variable,  $U(t) \in \mathfrak{R}^m$  is the external input,  $Y(t) \in \mathfrak{R}^m$  is the output,  $F$  and  $G$  are smooth vector fields and  $H$  is a smooth mapping. Without loss of generality, we suppose that the vector field  $F$  has at least one equilibrium point. The notion of passivity can be described as follows:

**Definition 1** (Byrnes et al., 1991)

If there exist a nonnegative constant  $\beta$  and a positive semi-definite function  $S(X(t))$  such that

$$\int_0^t U^T(\tau)Y(\tau)d\tau + \beta \geq \int_0^t S(X(\tau))d\tau, \quad \forall t \geq 0, \quad (3)$$

the system (1) - (2) is said to be passive from the external input  $U(t)$  to the output  $Y(t)$ .

The physical meaning of passive system is that the energy of the nonlinear system (1) - (2) can be increased only through the supply from the external source. In other words, a passive system cannot store more energy than it

is supplied. Passive system is naturally a stable system. Passive system exploits the input-output relationship based on energy-related considerations to analyze stability properties.

The following statement describes a basic stabilizability property of passive systems.

**Lemma 1** (Byrnes et al., 1991)

Suppose the system (1) - (2) is passive. Let  $\phi(\cdot)$  be any smooth function such that  $\phi(0) = 0$  and  $Y^T(t)\phi(Y(t)) > 0$  for each nonzero  $Y(t)$ . The control law  $U(t) = -\phi(Y(t))$  asymptotically stabilizes the equilibrium point of the system (1).

## PROBLEM FORMULATION

Consider a class of chaotic systems described by the following nonlinear differential equation:

$$\dot{x}(t) = Ax(t) + Bf(x(t)), \quad (4)$$

where  $x(t) \in \mathfrak{R}^n$  is the state vector,  $f(x(t)) \in \mathfrak{R}^n$  is the nonlinear function vector,  $A \in \mathfrak{R}^{n \times n}$  and  $B \in \mathfrak{R}^{n \times n}$  are known constant matrices. The system (4) is considered as a drive system and the response system with control input is introduced as follows:

$$\dot{z}(t) = Cz(t) + Dg(z(t)) + u(t) \quad (5)$$

where  $z(t) \in \mathfrak{R}^n$  and  $u(t) \in \mathfrak{R}^n$  are the state vector and the control input of the controlled response system, respectively.  $g(z(t)) \in \mathfrak{R}^n$  is the nonlinear function vector of the controlled response system,  $C \in \mathfrak{R}^{n \times n}$  and  $D \in \mathfrak{R}^{n \times n}$  are known constant matrices. The purpose of this paper is to design the feedback control input  $u(t)$  guaranteeing the passivity based synchronization. In order to design the feedback control input  $u(t)$ , we need information on states of drive and response systems. Thus, the control input  $u(t)$  in (5) depends on states of drive and response systems. Define the synchronization error  $e(t) = z(t) - x(t)$ . Then we obtain the synchronization error system.

$$\dot{e}(t) = (C+A)e(t) + Cx(t) - Ax(t) + Dg(z(t)) - Bf(x(t)) + u(t) \quad (6)$$

In this paper, we will use the passivity technique to synchronize two different chaotic systems. The controller proposed in this paper is based on an LMI. Using the passive method, it is very easy to prove the stability of the closed-loop system.

The following fact will be used for deriving the main result.

**Fact 1** (Schur complement)

Given constant symmetric matrices  $\Sigma_1, \Sigma_2, \Sigma_3$ , where  $\Sigma_1 = \Sigma_1^T$  and  $\Sigma_2 = \Sigma_2^T > 0$ , then  $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$  if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.$$

**MAIN RESULTS**

The LMI problem for achieving the passivity-based synchronization between two different chaotic systems is presented in the following theorem.

**Theorem 1**

For a given  $Q = Q^T > 0$ , if there exist  $X = X^T > 0$  and  $Y$  such that

$$\begin{bmatrix} (C+A)X + X(C+A)^T + Y + Y^T & X \\ X & -Q^{-1} \end{bmatrix} < 0, \quad (7)$$

then the error system (6), under the control input

$$u(t) = YX^{-1}e(t) - Cx(t) + Az(t) - Dg(z(t)) + Bf(x(t)) + v(t) \quad (8)$$

where  $v(t)$  is an external input signal, is passive from the external input signal  $v(t)$  to the output  $y(t)$  which is defined as

$$y(t) \cong 2Pe(t). \quad (9)$$

**Proof:** The closed-loop error system with the control input

$$\dot{\bar{u}}(t) = K\bar{e}(t) - Cx(t) + Az(t) - Dg(z(t)) + Bf(x(t)) + v(t)$$

where  $K \in \mathfrak{R}^{n \times n}$  is the gain matrix of the control input  $u(t)$ , can be written as

$$\dot{\bar{e}}(t) = [(C+A) + K]e(t) + v(t). \quad (10)$$

Consider a Lyapunov function

$$V(e(t)) = e^T(t)Pe(t) \quad (11)$$

where  $P = P^T > 0$ . Its time derivative along the trajectory of (10) is

$$\begin{aligned} \dot{V}(e(t)) &= \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) \\ &= e^T(t)[(C+A)^T P + P(C+A) + PK + K^T P]e(t) + 2e^T(t)Pv(t) \\ &= e^T(t)[(C+A)^T P + P(C+A) + PK + K^T P]e(t) + y^T(t)v(t). \end{aligned}$$

If the following matrix inequality is satisfied

$$(C+A)^T P + P(C+A) + PK + K^T P + Q < 0, \quad (12)$$

we have

$$\dot{V}(e(t)) < -e^T(t)Qe(t) + y^T(t)v(t). \quad (13)$$

Integrating both sides of (13) from 0 to  $t$  gives

$$V(e(t)) - V(e(0)) < -\int_0^t e^T(\tau)Qe(\tau)d\tau + \int_0^t y^T(\tau)v(\tau)d\tau. \quad (14)$$

Let  $\beta = V(e(0))$ . Since  $V(e(t)) \geq 0$ ,

$$\begin{aligned} \int_0^t y^T(\tau)v(\tau)d\tau + \beta &> \int_0^t e^T(\tau)Qe(\tau)d\tau + V(e(t)) \\ &\geq \int_0^t e^T(\tau)Qe(\tau)d\tau. \end{aligned} \quad (15)$$

The relation (15) satisfies the passivity definition (3). Therefore, the error system (6) is rendered to be passive from the external input signal  $v(t)$  to the output  $y(t)$  under the feedback control input  $\bar{u}(t) = K\bar{e}(t) - Cx(t) + Az(t) - Dg(z(t)) + Bf(x(t)) + v(t)$ .

From Fact 1, the matrix inequality (12) is equivalent to

$$\begin{bmatrix} (C+A)^T P + P(C+A) + PK + K^T P & I \\ I & -Q^{-1} \end{bmatrix} < 0. \quad (16)$$

Pre- and post-multiplying (16) by  $\text{diag}(P^{-1}, I)$  and introducing change of variables such as  $X = P^{-1}$  and  $Y = KP^{-1}$ , (16) is equivalently changed into the LMI (7). Then the gain matrix of the control input  $u(t)$  is given by  $K = YX^{-1}$ . This completes the proof.

**Corollary 1** (zero-input error response)

If the external input signal  $v(t)$  is zero, the closed-loop

error system is asymptotically stable.

**Proof:** When  $v(t) = 0$ , we obtain

$$\dot{V}(e(t)) < -e^T(t)Qe(t) \leq 0 \tag{17}$$

from (13). This guarantees

$$\lim_{t \rightarrow \infty} e(t) = 0 \tag{18}$$

from Lyapunov stability theory. This completes the proof.

According to Lemma 1, once the error system (6) has been rendered passive, the external input signal  $v(t) = -\phi(y(t))$  satisfying  $\phi(0) = 0$  and  $y^T(t)\phi(y(t)) > 0$  for each nonzero  $y(t)$  asymptotically stabilizes the error system (6). For example, a pure gain output feedback  $v(t) = -\mu y(t)$  ( $\mu > 0$ ) can stabilize the error system (6).

**Corollary 2** (nonzero-input error response)

If the external input signal  $v(t)$  is selected as

$$v(t) = -\mu y(t) = -2\mu P e(t), \quad \mu > 0, \tag{19}$$

the closed-loop error system is asymptotically stable.

**Proof:** For  $v(t) = -\mu y(t)$ , the time derivative of  $V(e(t))$  satisfies

$$\dot{V}(e(t)) < -e^T(t)Qe(t) - \mu y^T(t)y(t) \leq 0 \tag{20}$$

From (13). This guarantees the asymptotical stability from Lyapunov stability theory. This completes the proof.

**Remark 1**

Various efficient convex optimization algorithms can be used to check whether the LMI (7) is feasible. In this paper, in order to solve the LMI, we utilize MATLAB LMI Control Toolbox (Gahinet, Nemirovski, Laub, and Chilali, 1995), which implements state-of the art interior-point algorithms.

**NUMERICAL EXAMPLE**

In this section, to verify and demonstrate the

$$X = \begin{bmatrix} 0.3301 & 0.0000 & 0.0000 \\ 0.0000 & 0.3301 & 0.0000 \\ 0.0000 & 0.0000 & 0.3301 \end{bmatrix}, Y = \begin{bmatrix} -0.4951 & -26.2042 & 2.3637 \\ 25.8741 & -0.5611 & 0.5437 \\ -0.0532 & 0.0933 & 1.7824 \end{bmatrix}$$

effectiveness of the proposed method, we discuss the

simulation result for synchronizing Rossler system (Rössler, 1976) and Genesio-Tesi system (Genesio and Tesi, 1992). Consider the following Rossler chaotic system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & b & 0 \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a + x_1(t)x_3(t) \end{bmatrix}, \tag{21}$$

where  $x_i(t)$  ( $i = 1,2,3$ ) is the state variable of the Rossler system. The Rossler system exhibits chaotic behavior when the system parameters are chosen as  $a = 0.2$ ,  $b = 0.2$  and  $c = 5.7$ .

Now consider the following Genesio-Tesi system as the controlled response system:

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \dot{z}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c_1 & -b_1 & -a_1 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z_1^2(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}, \tag{22}$$

where  $z_i(t)$  ( $i = 1,2,3$ ) and  $u_i(t)$  ( $i = 1,2,3$ ) are the state variable and the control input of the controlled Genesio-Tesi system, respectively. This system is chaotic for the system parameters  $a_1 = 1.2$ ,  $b_1 = 2.92$  and  $c_1 = 6$ . The Genesio-Tesi system, proposed by Genesio and Tesi, is one of paradigms of chaos since it captures many features of chaotic systems. Applying Theorem 1 with  $Q = 5 \times I$ , where  $I \in R^{3 \times 3}$  is an identity matrix, yields

$$X = \begin{bmatrix} 0.3301 & 0.0000 & 0.0000 \\ 0.0000 & 0.3301 & 0.0000 \\ 0.0000 & 0.0000 & 0.3301 \end{bmatrix}, Y = \begin{bmatrix} -0.4951 & -26.2042 & 2.3637 \\ 25.8741 & -0.5611 & 0.5437 \\ -0.0532 & 0.0933 & 1.7824 \end{bmatrix}, \tag{23}$$

Figure 1 shows state trajectories for drive and response systems when the initial conditions are given by

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix}. \tag{24}$$

From Figure 1, it can be seen that drive and response systems are indeed achieving chaos synchronization. The simulation result in Figure 2 shows synchronization error trajectories for different values of the parameter  $\mu$  in the external input signal  $v(t)$  (19). It is found from Figure 2 that the parameter  $\mu$  only influences the transient response and the bigger  $\mu$  gives the better response.

**Conclusion**

In this paper, we propose a new passivity-based

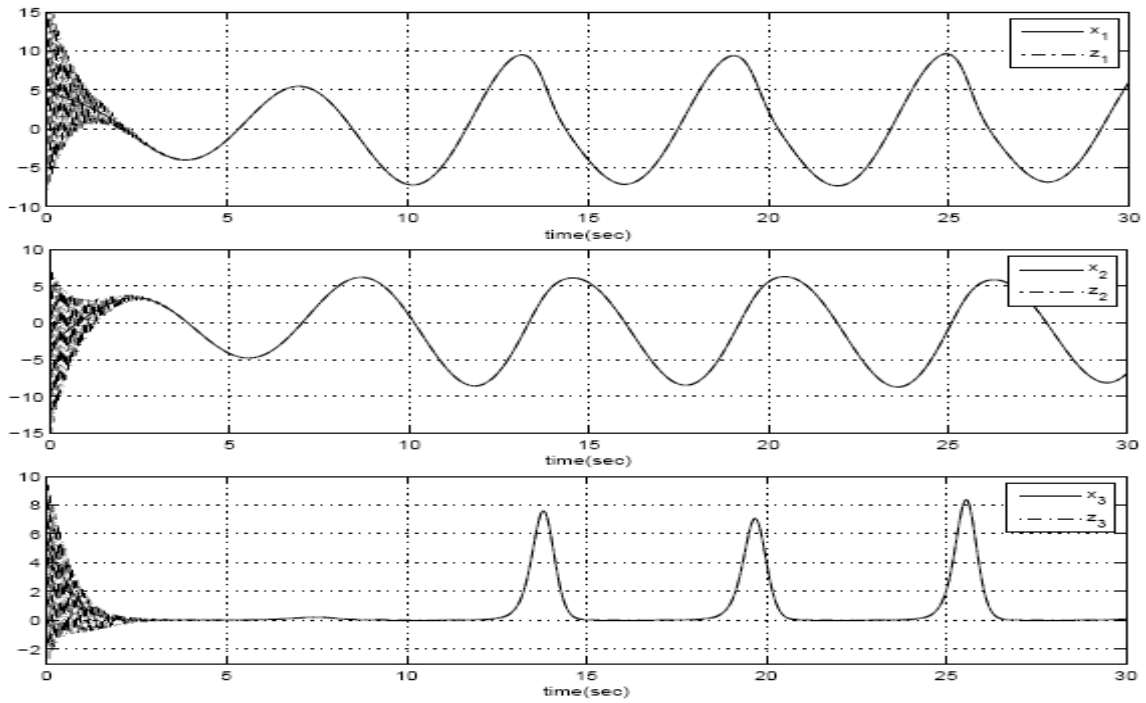


Figure 1. State trajectories.

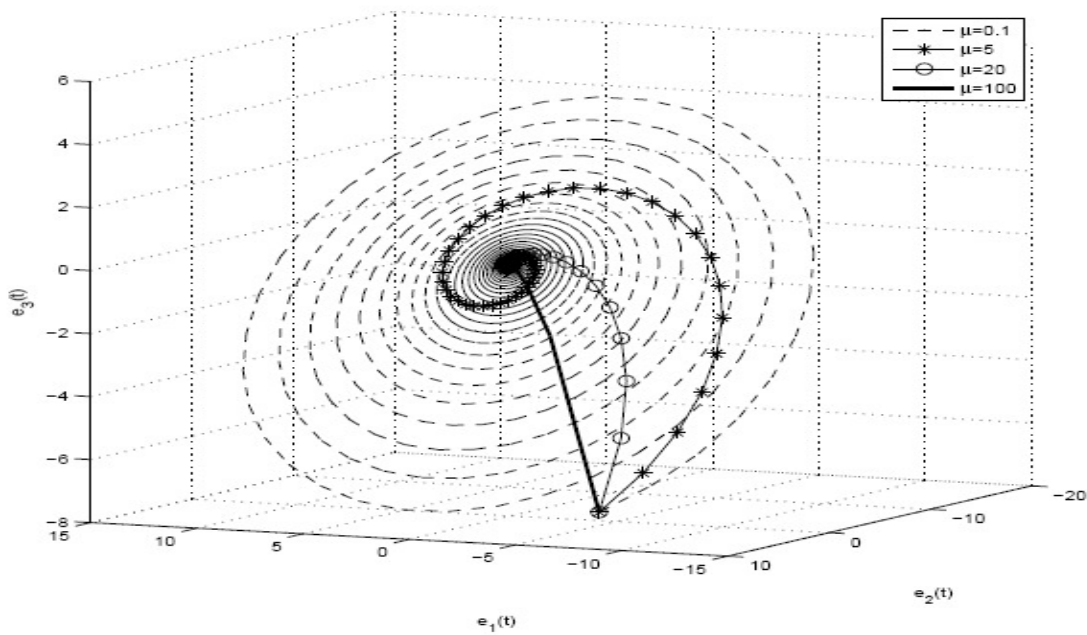


Figure 2. Synchronization error trajectories.

synchronization method for two different chaotic systems. Based on Lyapunov stability theory and LMI approach, the proposed scheme guarantees to make the closed-loop error system passive and also asymptotically stable. Furthermore, the synchronization between the Rossler system and the Genesio-Tesi system is given to

illustrate the effectiveness of the proposed scheme.

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