International Journal of Physical Sciences Vol. 7(9), pp. 1333 - 1345, 23 February, 2012
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS11.1507
ISSN 1992 - 1950 ©2012 Academic Journals

Full Length Research Paper

Flow of a binary mixture of fluids in a semicircular duct

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Accepted 31 January, 2012

The purpose of this work is to examine the flow of a binary mixture including chemically inert incompressible Newtonian fluids in a duct of semicircular cross-section. Such a flow model has great significance not only of its own theoretical interest, but also for application to various engineering processes. The governing equations have been solved analytically using the finite Fourier sine and Hankel transforms for the following four problems: (1) steady Couette flow in a semicircular duct, (2) unsteady Couette flow in a semicircular duct, (3) steady Poiseuille flow in a semicircular duct, (4) unsteady Poiseuille flow in a semicircular duct. The previous solutions corresponding to pure Newtonian fluid appear as the special cases of the present analysis.

Key words: Binary mixture, Newtonian fluid, Couette flow, Poiseuille flow, Fourier sine transform, Hankel transform.

INTRODUCTION

The subject of a mixture of fluids is currently one of importance in view of its relevance to a number of engineering processes. A familiar example is an emulsion, which is the dispersion of one fluid within another fluid. Typical emulsions are produced by mixing water and oil with an emulsifier. Water by itself is a very poor lubricant, but when mixed with oils to form emulsions, some useful lubricants can be developed. These liquids are used as coolants in metalworking where the combination of the lubricity of oil, high conductivity and the latent heat of water provide the optimum fluid for this application. Mining machinery is also lubricated by water-based fluids to minimize the risk of fire from leakage of lubricants. Another example where the fluid mixtures play an important role is in multigrade oils. In order to enhance the lubrication properties of mineral oils such as the viscosity index, polymeric type fluids are added to the base oil (Al-Sharif et al., 1993; Chamniprasart et al., 1993; Wang et al., 1993; Dai and Khonsari, 1994; Stachowiak and Batchelor, 2001).

Truesdell (1957) was the first to derive the balance and conservation equations through the use of a continuum theory of mixtures. The principal idea for the theoretical treatment of the mechanics and thermodynamics of mixtures is the supposition that the mutual interconnection of the different constituents is conceptually idealized so as to assume that each spatial point is simultaneously occupied by one particle from each constituent (Truesdell and Toupin, 1960). Such an idealization obviously requires the mixture to be sufficiently dense. This principle forms the basis of approach in most texts and articles dealing with the theory of mixtures. Truesdell's (1957) pioneering work gave impetus to many researches on the continuum theory of mixtures, and a good amount of literature has grown up around this subject. The theoretical progress and detailed analysis of various results on the mixture theory can be found in the review articles by Bowen (1976), Atkin and Craine (1976a, b), Bedford and Drumheller (1983) and in the books by Truesdell (1984), Samohyl (1987) and Rajagopal and Tao (1995). Much has been written since 1957 on the subject of the theory of mixtures, and they dealt with the general formulations of the basic equations and constitutive models. A literature survey clearly indicates that very little work seems to have been done on applications of the theory of interacting continua to practical problems. This is because there are serious difficulties with regard to specifying boundary conditions and modeling of the interactions between constituents in mixture theory (Rajagopal and Tao, 1995; Massoudi, 2003). The mixture

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of two compressible Newtonian fluids was first considered by Green and Naghdi (1965). Their results were used by Mills (1966) studying a binary mixture of incompressible Newtonian fluids and applied to problem of helical flow. Craine (1971) examined the flow induced by the steady oscillations of an infinite plate in a mixture of two incompressible Newtonian fluids. In his subsequent study (Craine, 1973), he considered the same problem for a binary mixture of incompressible Newtonian hemihedral fluids. Wilhelm and Van Der Werff (1977) presented a theoretical analysis for the flows of two miscible, viscous, incompressible fluids, subject to oscillatory pressure gradients in a cylindrical tube. Later, some exact solutions for the flow of a binary mixture of incompressible Newtonian fluids were obtained by Göğüş (1988, 1991, 1992a, b, 1994, 1995). Several problems relating to the mechanics of oil and water emulsions were considered within the context of the mixture theory by Al-Sharif et al. (1993), Chamniprasart et al. (1993) and Wang et al. (1993). Recently, Barış (2005) considered the unsteady flow of two chemically inert incompressible Newtonian fluids in the annular region between two infinitely long coaxial cylinders.

To the best of the authors' knowledge, no results (theoretical or experimental) currently exist for the steady and unsteady flows of a binary mixture of incompressible Newtonian fluids in semicircular ducts. The objective of the present investigation is to generate theoretical results for the flow of the mixture of two incompressible Newtonian fluids in a semicircular duct. Therefore, this paper has not been concerned with experimental substantiating the validity of the established analytical solutions. The flow and heat transfer in these ducts, apart from their theoretical interest, are of considerable practical importance and arise frequently in industrial processes. For example, in response to the strong social demand for reduction of fuel oil consumption, a semicircular duct type energy saving device for a full ship has been developed. It is found that the energy saving effect of this device was greater than that of conventional circular ducts, and horsepower was reduced by approximately 5% based on a model test (Yasuhiko et al., 2007). Besides, semicircular ducts are also used in heat exchanger design in various engineering applications. These ducts can carry Newtonian or non-Newtonian fluid under the constant wall temperature or constant heat flux boundary conditions (Oztop, 2005). In the present paper, we have sought special semi inverse solutions of the equations of motion governing the steady and unsteady flows of a binary mixture of incompressible Newtonian fluids in semicircular ducts. We have obtained the exact solutions in series form for the velocity fields by means of finite integral transforms. Exact solutions are important not only because they are analytical solutions for some fundamental flows but also because they can serve as accuracy checks for experimental, numerical and asymptotic methods.

FUNDAMENTAL EQUATIONS

In this section, for the sake of completeness and continuity, we provide a very brief summary of the basic balance laws and the appropriate constitutive theory for a binary mixture of incompressible Newtonian fluids. For a review of these issues, the reader is referred to the articles by Atkin and Craine (1976a, b).

We considered a mixture of two continua $S^{(1)}$ and $S^{(2)}$ which are in motion relative to each other. We assume that each point within the mixture is occupied simultaneously by $S^{(1)}$ and $S^{(2)}$ and refer the motion of the continua to a fixed system of rectangular Cartesian coordinates. Let $\mathbf{X}^{(\beta)}$ be the reference position of typical particles of the β th constituent. Throughout this paper, β takes the values 1 and 2. The motion of the β th constituent is denoted by:

$$\mathbf{x}^{(\beta)} = \mathbf{x}^{(\beta)}(\mathbf{X}^{(\beta)}, t) \tag{1}$$

We shall assume this motion is one-to-one, continuous and invertible. All fields appearing in the subsequent equations are regarded as being continuous functions of position and time. We shall denote the velocity vector associated with the motion through

$$\mathbf{v}^{(\beta)} = \frac{D\mathbf{x}^{(\beta)}}{Dt} \tag{2}$$

where D/Dt is the convective time derivative. The deformation-rate and spin tensors are given, respectively by:

$$2d_{ij}^{(\beta)} = v_{i,j}^{(\beta)} + v_{j,i}^{(\beta)}, \quad 2w_{ij}^{(\beta)} = v_{i,j}^{(\beta)} - v_{j,i}^{(\beta)} \tag{3}$$

where a comma denotes partial differentiation with respect to $x_k^{(\beta)}$.
We define the mean velocity of the mixture $\mathbf{w}$ and the total density of the mixture ρ by the equations as follows:

$$\rho w_i = \rho_1 v_i^{(1)} + \rho_2 v_i^{(2)} \tag{4}$$

$$\rho = \rho_1 + \rho_2 \tag{5}$$

in which ρ_1 and ρ_2 are the densities of $S^{(1)}$ and $S^{(2)}$ at time t, measured per unit volume of mixture.
If thermal, chemical and electro-magnetic effects are not considered, the fundamental laws of continuum mechanics reduce to the conservation equations for mass, linear momentum and angular momentum. The balance of angular momentum for the mixture results in the symmetry of the total stress tensor $\boldsymbol{\sigma}$ for mixture, though the balance of angular momentum for $S^{(\beta)}$ shows that the partial stress tensor $\boldsymbol{\sigma}^{(\beta)}$ need not to be symmetric. The conservation of mass and the conservation of linear momentum for a binary mixture are as follows:

$$\frac{\partial \rho_1}{\partial t} + (\rho_1 v_i^{(1)})_{,i} = 0, \quad \frac{\partial \rho_2}{\partial t} + (\rho_2 v_i^{(2)})_{,i} = 0 \tag{6}$$

$$\rho_1 \frac{D^{(1)}v_k^{(1)}}{Dt} = \sigma_{ik,i}^{(1)} - f_k + \rho_1 F_k^{(1)}, \quad \rho_2 \frac{D^{(2)}v_k^{(2)}}{Dt} = \sigma_{ik,i}^{(2)} + f_k + \rho_2 F_k^{(2)} \quad (7)$$

where f_k, $\sigma_{ik}^{(\beta)}$ and $F_k^{(\beta)}$ are in turn the mechanical interaction (local exchange of momentum) between the two components, partial stress and body force per unit mass of the β th constituent. The interaction force f_k is possibly the most important of all interaction terms. The paper by Massoudi (2003) discussed a variety of possible form of this term. For instance, for fluid-solid and fluid-fluid mixtures, in general, this interaction force depends on densities, temperatures, velocity differences, their gradients and possibly other quantities. Such interactions play a very important role in the nature of the solutions (Johnson et al., 1991a, b). In this study, we assume that the interaction force incorporates only the effect of drag and depends on the velocity differences in a linear fashion. Calculations based on this assumption for various problems related to binary fluid mixtures have been carried out by some authors like Craine (1971, 1973), Al-Sharif et al. (1993), Chamniprasart et al. (1993), Wang et al. (1993), Göğüş (1988, 1991, 1992a, b, 1994, 1995) and Barış (2005).

In this work, we shall concern ourselves with a mixture of two incompressible Newtonian fluids. In the reference state before mixing, let the density of $S^{(\beta)}$ be $\rho_{\beta 0}$ which a constant in view of the assumed incompressibility is. Introducing a volume fraction ϕ_1, defined as the proportion by volume of $S^{(1)}$ and assuming that the mixture does not contain voids, it follows that the densities of $S^{(1)}$ and $S^{(2)}$ at time t are given by:

$$\rho_1 = \phi_1 \rho_{10}, \quad \rho_2 = (1-\phi_1)\rho_{20} \quad (8)$$

And hence,

$$\frac{\rho_1}{\rho_{10}} + \frac{\rho_2}{\rho_{20}} = 1 \quad (9)$$

Using Equations 5 and 9, it can be easily shown that

$$\rho_1 = \frac{\rho_{10}(\rho_{20}-\rho)}{\rho_{20}-\rho_{10}}, \quad \rho_2 = \frac{\rho_{20}(\rho-\rho_{10})}{\rho_{20}-\rho_{10}} \quad (10)$$

Substituting Equation 10 into Equation 6 and eliminating $\partial\rho/\partial t$ between the resulting equations, we get

$$(\rho_{20}-\rho)d_{ii}^{(1)} + (\rho-\rho_{10})d_{ii}^{(2)} - \rho_{,i}(v_i^{(1)} - v_i^{(2)}) = 0 \quad (11)$$

Looking at Equation 7, it is clear that in order to close the system of equations, we need to provide constitutive relations for $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\sigma}^{(2)}$ and $\mathbf{f}$. The derivation of the constitutive equations appropriate to a mixture of two incompressible Newtonian fluids has been outlined in Atkin and Craine (1976a, b). If the mixture is considered to be purely mechanical system; that is, thermal effects are ignored, the relevant equations are:

$$A_\beta = A_\beta(\rho), \quad A = A(\rho) \quad (12)$$

$$p_1 = (\rho - \rho_{20})\left(\rho_1 \frac{dA_1}{d\rho} + \lambda\right), \quad p_2 = (\rho - \rho_{10})\left(\rho_2 \frac{dA_2}{d\rho} - \lambda\right) \quad (13)$$

$$f_k = \alpha(v_k^{(1)} - v_k^{(2)}) - \lambda\rho_{,k} \quad (14)$$

$$\sigma_{ik}^{(1)} = (-p_1 + \lambda_1 d_{jj}^{(1)} + \lambda_3 d_{jj}^{(2)})\delta_{ik} + 2\mu_1 d_{ik}^{(1)} + 2\mu_3 d_{ik}^{(2)} + \lambda_5 (w_{ik}^{(1)} - w_{ik}^{(2)}) \quad (15)$$

$$\sigma_{ik}^{(2)} = (-p_2 + \lambda_4 d_{jj}^{(1)} + \lambda_2 d_{jj}^{(2)})\delta_{ik} + 2\mu_4 d_{ik}^{(1)} + 2\mu_2 d_{ik}^{(2)} - \lambda_5 (w_{ik}^{(1)} - w_{ik}^{(2)}) \quad (16)$$

where A_β denotes the partial Helmholtz free energy of the β th constituent, λ is a lagrange multiplier associated with the constraint of Equation 11 and the Helmholtz free energy of the mixture A is defined by:

$$\rho A = \rho_1 A_1 + \rho_2 A_2 \quad (17)$$

And the coefficient $\alpha, \lambda_1, ..., \lambda_5, \mu_1, ..., \mu_4$ satisfy the inequalities

$$\alpha \geq 0,\ \lambda_5 \geq 0,\ \mu_1 \geq 0,\ \mu_2 \geq 0,\ \lambda_1 + \frac{2}{3}\mu_1 \geq 0,\ \lambda_2 + \frac{2}{3}\mu_2 \geq 0,\ (\mu_3+\mu_4)^2 \leq 4\mu_1\mu_2,$$

$$\left[\lambda_3 + \lambda_4 + \frac{2}{3}(\mu_3+\mu_4)\right]^2 \leq 4\left(\lambda_1 + \frac{2}{3}\mu_1\right)\left(\lambda_2 + \frac{2}{3}\mu_2\right) \quad (18)$$

Finally, neglecting the body forces, we derived the equations governing the flow of a mixture of two incompressible Newtonian fluids. For this purpose, inserting f_k, $\sigma_{ik}^{(1)}$ and $\sigma_{ik}^{(2)}$ from Equations 14 to 16 into Equation 7, with the help of Equation 3, one gets the following equations of motion:

$$\rho_1 \frac{D^{(1)}v_k^{(1)}}{Dt} + p_{1,k} - \lambda\rho_{,k} = M_1 v_{k,ii}^{(1)} + M_2 v_{k,ii}^{(2)} + M_5 v_{i,ik}^{(1)} + M_6 v_{i,ik}^{(2)} + v_{i,i}^{(1)}\lambda_{1,k}$$
$$+ v_{i,i}^{(2)}\lambda_{3,k} + v_{k,i}^{(1)}M_{1,i} + v_{k,i}^{(2)}M_{2,i} + v_{i,k}^{(1)}M_{9,i} + v_{i,k}^{(2)}M_{10,i} - \alpha(v_k^{(1)} - v_k^{(2)}) \quad (19)$$

$$\rho_2 \frac{D^{(2)}v_k^{(2)}}{Dt} + p_{2,k} + \lambda\rho_{,k} = M_3 v_{k,ii}^{(1)} + M_4 v_{k,ii}^{(2)} + M_7 v_{i,ik}^{(1)} + M_8 v_{i,ik}^{(2)} + v_{i,i}^{(1)}\lambda_{4,k}$$
$$+ v_{i,i}^{(2)}\lambda_{2,k} + v_{k,i}^{(1)}M_{3,i} + v_{k,i}^{(2)}M_{4,i} + v_{i,k}^{(1)}M_{11,i} + v_{i,k}^{(2)}M_{12,i} + \alpha(v_k^{(1)} - v_k^{(2)}) \quad (20)$$

where

$$M_1 = \mu_1 - \frac{\lambda_5}{2},\ M_2 = \mu_3 + \frac{\lambda_5}{2},\ M_3 = \mu_4 + \frac{\lambda_5}{2},\ M_4 = \mu_2 - \frac{\lambda_5}{2},\ M_5 = \lambda_1 + \mu_1 + \frac{\lambda_5}{2},$$

$$M_6 = \lambda_3 + \mu_3 - \frac{\lambda_5}{2},\quad M_7 = \lambda_4 + \mu_4 - \frac{\lambda_5}{2},\quad M_8 = \lambda_2 + \mu_2 + \frac{\lambda_5}{2},\quad M_9 = \mu_1 + \frac{\lambda_5}{2},$$

$$M_{10} = \mu_3 - \frac{\lambda_5}{2},\quad M_{11} = \mu_4 - \frac{\lambda_5}{2},\quad M_{12} = \mu_2 + \frac{\lambda_5}{2} \quad (21)$$

Note that, under isothermal conditions, the material coefficients M_1

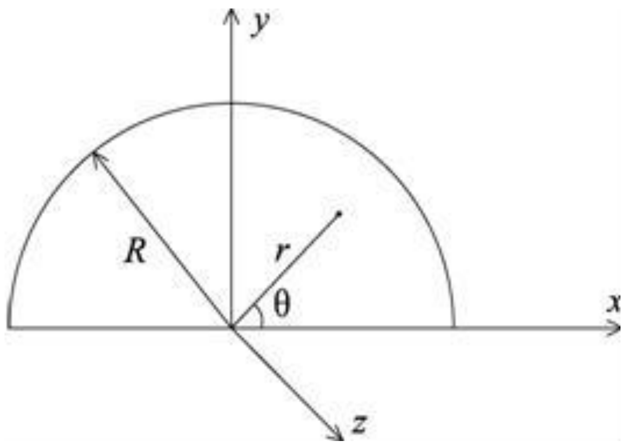

Figure 1. Sketch of flow geometry and coordinate system.

etc. appearing in Equations 19 and 20 depend only on the total density ρ.

STEADY COUETTE FLOW IN A SEMICIRCULAR DUCT

Here, we begin with the case of the fully developed stage of the flow of a binary mixture of incompressible Newtonian fluids in a duct of semicircular cross-section. The flow geometry and the coordinate system are shown in Figure 1. It is assumed that the flow is entirely driven by the motion of the bottom wall with steady velocity U in the z - direction in the absence of the pressure gradient $\partial p/\partial z$.

We seek solutions in which the velocity vector of the β th fluid and densities are assumed to have the form:

$$\mathbf{v}^{(\beta)} = \left\{0, 0, w_{\beta SC}(r,\theta)\right\}, \quad \rho_1 = \rho_1(r,\theta), \quad \rho_2 = \rho_2(r,\theta) \quad (22)$$

where $w_{\beta SC}$ is the z -component of the velocity vector of the β th fluid. With this assumption, it is shown that the equations of continuity, that is, Equation 6 can be satisfied identically. Substituting Equation 22 into the r - and θ- components of the equations of motion, that is, Equations 19 and 20, we get:

$$\frac{\partial p_1}{\partial r} = \lambda \frac{\partial \rho}{\partial r}, \quad \frac{\partial p_2}{\partial r} = -\lambda \frac{\partial \rho}{\partial r} \quad (23)$$

$$\frac{\partial p_1}{\partial \theta} = \lambda \frac{\partial \rho}{\partial \theta}, \quad \frac{\partial p_2}{\partial \theta} = -\lambda \frac{\partial \rho}{\partial \theta} \quad (24)$$

With the use of Equations 10, 13 and 17, elimination of $\partial\lambda/\partial r$ between Equations 23_1 and
23_2, and that of $\partial\lambda/\partial\theta$ between Equations 24_1 and 24_2 give, respectively,

$$(\rho-\rho_{10})(\rho-\rho_{20})\,\frac{\partial \rho}{\partial r}\,\frac{d^2(\rho A)}{d\rho^2} = 0\,, \quad (25)$$

$$(\rho-\rho_{10})(\rho-\rho_{20})\,\frac{\partial \rho}{\partial \theta}\,\frac{d^2(\rho A)}{d\rho^2} = 0\,, \quad (26)$$

and since, in general, $\rho \neq \rho_{10}$, $\rho \neq \rho_{20}$ and $d^2(\rho A)/d\rho^2 \neq 0$ we deduce that $\rho = \rho_0 = const.$ As a result, the constitutive coefficients M_1 etc. in Equations 19 and 20 are constants. In the light of this arguments, the z - components of the equations of motions reduce to

$$M_1\left(\frac{\partial^2 w_{1SC}}{\partial r^2} + \frac{1}{r}\frac{\partial w_{1SC}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_{1SC}}{\partial \theta^2}\right) + M_2\left(\frac{\partial^2 w_{2SC}}{\partial r^2} + \frac{1}{r}\frac{\partial w_{2SC}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_{2SC}}{\partial \theta^2}\right)$$
$$-\alpha(w_{1SC} - w_{2SC}) = 0 \quad (27)$$

$$M_3\left(\frac{\partial^2 w_{1SC}}{\partial r^2} + \frac{1}{r}\frac{\partial w_{1SC}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_{1SC}}{\partial \theta^2}\right) + M_4\left(\frac{\partial^2 w_{2SC}}{\partial r^2} + \frac{1}{r}\frac{\partial w_{2SC}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_{2SC}}{\partial \theta^2}\right)$$
$$+\alpha(w_{1SC} - w_{2SC}) = 0 \quad (28)$$

It is convenient at this point to introduce dimensionless variables and material constants. If $\overline{f}$ is used to denote the dimensionless form of a quantity f, it follows that:

$$\overline{M}_i = \frac{M_i}{\mu}, \quad \overline{\alpha} = \frac{\alpha R^2}{\mu}, \quad \overline{w}_{\beta SC} = \frac{w_{\beta SC}}{U}, \quad \overline{r} = \frac{r}{R} \quad (29)$$

where μ is viscosity coefficient of the mixture. The dimensionless governing equations are obtained from Equations 27 and 28 by replacing variables and material constants by those given in Equation 29, so they are not rewritten here.

The boundary condition for the velocity fields are:

$$\overline{w}_{\beta SC}(1,\theta) = 0, \quad \overline{w}_{\beta SC}(\overline{r},0) = 1, \quad \overline{w}_{\beta SC}(\overline{r},\pi) = 1\,. \quad (30)$$

Throughout this paper, henceforth for convenience, unless stated otherwise, there is the drop of the bars that appear over the dimensionless quantities.

We assume a solution in the form

$$w_{\beta SC}(r,\theta) = 1 - f_\beta(r,\theta)\,. \quad (31)$$

Substitution of $w_{\beta SC}(r,\theta)$ in Equations 27 and 28 yields

$$M_1\left(\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f_1}{\partial \theta^2}\right) + M_2\left(\frac{\partial^2 f_2}{\partial r^2} + \frac{1}{r}\frac{\partial f_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f_2}{\partial \theta^2}\right) - \alpha(f_1 - f_2) = 0 \quad (32)$$

$$M_3\left(\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f_1}{\partial \theta^2}\right) + M_4\left(\frac{\partial^2 f_2}{\partial r^2} + \frac{1}{r}\frac{\partial f_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f_2}{\partial \theta^2}\right) + \alpha(f_1 - f_2) = 0 \quad (33)$$

In a similar manner, if $w_{\beta SC}(r,\theta)$ is inserted in the boundary conditions of Equation 30, we obtain

$$f_\beta(1,\theta) = 1, \quad f_\beta(r,0) = 0, \quad f_\beta(r,\pi) = 0\,. \quad (34)$$

Finite Fourier sine transform was used to solve the two stated simultaneous partial differential equations with the boundary conditions of Equation 34. The finite Fourier sine transform of a function $f(\theta)$ defined for $0<\theta<\pi$ is (Debnath, 1995):

$$F\{f(\theta);\theta\to k\}=\tilde{f}(k)=\int_0^{\pi} f(\theta)\sin(k\theta)d\theta, \qquad k=1,2,3,... \quad (35)$$

with inverse transform

$$F^{-1}\{\tilde{f}(k);k\to\theta\}=f(\theta)=\frac{2}{\pi}\sum_{k=1}^{\infty}\tilde{f}(k)\sin(k\theta) \quad (36)$$

Application of the finite Fourier sine transform to Equations 32 and 33 with respect to θ, taking Equations 34_2 and 34_3 into account, gives:

$$M_1\left(\frac{d^2\tilde{f}_1(r,k)}{dr^2}+\frac{1}{r}\frac{d\tilde{f}_1(r,k)}{dr}-\frac{k^2}{r^2}\tilde{f}_1(r,k)\right)+M_2\left(\frac{d^2\tilde{f}_2(r,k)}{dr^2}+\frac{1}{r}\frac{d\tilde{f}_2(r,k)}{dr}-\frac{k^2}{r^2}\tilde{f}_2(r,k)\right)$$
$$-\alpha[\tilde{f}_1(r,k)-\tilde{f}_2(r,k)]=0\,. \quad (37)$$

$$M_3\left(\frac{d^2\tilde{f}_1(r,k)}{dr^2}+\frac{1}{r}\frac{d\tilde{f}_1(r,k)}{dr}-\frac{k^2}{r^2}\tilde{f}_1(r,k)\right)+M_4\left(\frac{d^2\tilde{f}_2(r,k)}{dr^2}+\frac{1}{r}\frac{d\tilde{f}_2(r,k)}{dr}-\frac{k^2}{r^2}\tilde{f}_2(r,k)\right)$$
$$+\alpha[\tilde{f}_1(r,k)-\tilde{f}_2(r,k)]=0\,, \quad (38)$$

subject to the transform of Equation 34_1

$$\tilde{f}_\beta(1,k)=\frac{[1-(-1)^k]}{k}, \qquad k=1,2,3,... \quad (39)$$

Subtracting M_4 times Equation 37 from M_2 times Equation 38, and M_3 times Equation 37 from M_1 times Equation 38, we get the following equations, respectively

$$n_1\left(\frac{d^2\tilde{f}_1}{dr^2}+\frac{1}{r}\frac{d\tilde{f}_1}{dr}-\frac{k^2}{r^2}\tilde{f}_1\right)-\alpha(\tilde{f}_1-\tilde{f}_2)(M_2+M_4)=0 \quad (40)$$

$$-n_1\left(\frac{d^2\tilde{f}_2}{dr^2}+\frac{1}{r}\frac{d\tilde{f}_2}{dr}-\frac{k^2}{r^2}\tilde{f}_2\right)-\alpha(\tilde{f}_1-\tilde{f}_2)(M_1+M_3)=0 \quad (41)$$

and the sum of the Equations 40 and 41 is:

$$n_1\left(\frac{d^2}{dr^2}(\tilde{f}_1-\tilde{f}_2)+\frac{1}{r}\frac{d}{dr}(\tilde{f}_1-\tilde{f}_2)-\frac{k^2}{r^2}(\tilde{f}_1-\tilde{f}_2)\right)-n_2(\tilde{f}_1-\tilde{f}_2)=0 \quad (42)$$

where $n_1=M_1M_4-M_2M_3$ and $n_2=\alpha(M_1+M_2+M_3+M_4)$. It is clear that the solution of Equation 42 which satisfies the boundary condition $\tilde{f}_1(1,k)-\tilde{f}_2(1,k)=0$ is

$$\tilde{f}_1-\tilde{f}_2=0\,. \quad (43)$$

Substituting Equation 43 into Equations 40 and 41, and solving them under the boundary condition of Equation 39, we have

$$\tilde{f}_\beta=\frac{[1-(-1)^k]}{k}r^k\,. \quad (44)$$

With the help of Equation 36, inverting Equation 44 and then substituting of the results into Equation 31, we find

$$w_{\beta SC}(r,\theta)=1-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{[1-(-1)^k]}{k}r^k\sin(k\theta)\,. \quad (45)$$

It is recorded that there is no relative motion between the mixture constituents. For steady flows, if both gravitational effects and applied pressure gradients are absent, the fluids in a mixture will have the same velocities (Atkin and Crain, 1976b). Moreover, the velocity field is identical to that resulting from the Navier-Stokes theory.

UNSTEADY COUETTE FLOW IN A SEMICIRCULAR DUCT

Now is the time to examine unsteady Couette flow of a mixture of two incompressible Newtonian fluids in a semicircular duct. The mixture and the walls of the duct are initially at rest. The bottom wall is suddenly accelerated from rest and moves in its own plane with a constant velocity U. It is assumed that the flow is caused by the motion of the bottom wall, the pressure far upstream and downstream being kept equal throughout the motion. Thus, the pressure gradient in the z-direction are zero.

It seems reasonable to assume that the velocity distribution and total density in cylindrical coordinates are of the form

$$\mathbf{v}^{(\beta)}=\{0,0,w_{\beta C}(r,\theta,t)\}, \quad \rho_1=\rho_1(r,\theta,t), \quad \rho_2=\rho_2(r,\theta,t) \quad (46)$$

Substitution of Equation 46 into Equations 6 gives:

$$\frac{\partial\rho_1}{\partial t}=0, \quad \frac{\partial\rho_2}{\partial t}=0, \quad (47)$$

Thus, $\partial\rho/\partial t=0$ and $\rho=\rho(r,\theta)$. As made in the preceding section, elimination of $\partial\lambda/\partial r$ between r-components of the equations of motion, and that of $\partial\lambda/\partial\theta$ between the θ-components of the equations of motion give, respectively Equations 25 and 26, which imply that ρ is a constant. Since ρ has been proved to be a constant, all of the material coefficients in Equations 19 and 20 are constants. As a result, the dimensionless equation of motion in the z-direction are as follows:

$$\bar{M}_1\left(\frac{\partial^2\bar{w}_{1C}}{\partial\bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial\bar{w}_{1C}}{\partial\bar{r}}+\frac{1}{\bar{r}^2}\frac{\partial^2\bar{w}_{1C}}{\partial\theta^2}\right)+\bar{M}_2\left(\frac{\partial^2\bar{w}_{2C}}{\partial\bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial\bar{w}_{2C}}{\partial\bar{r}}+\frac{\partial^2\bar{w}_{2C}}{\partial\theta^2}\right)$$
$$-\bar{\alpha}(\bar{w}_{1C}-\bar{w}_{2C})=\bar{\rho}_1\frac{\partial\bar{w}_{1C}}{\partial\bar{t}}\,, \quad (48)$$

$$\bar{M}_3\left(\frac{\partial^2\bar{w}_{1C}}{\partial\bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial\bar{w}_{1C}}{\partial\bar{r}}+\frac{1}{\bar{r}^2}\frac{\partial^2\bar{w}_{1C}}{\partial\theta^2}\right)+\bar{M}_4\left(\frac{\partial^2\bar{w}_{2C}}{\partial\bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial\bar{w}_{2C}}{\partial\bar{r}}+\frac{\partial^2\bar{w}_{2C}}{\partial\theta^2}\right)$$

$$+\bar{\alpha}(\bar{w}_{1C}-\bar{w}_{2C})=\bar{\rho}_2\frac{\partial \bar{w}_{2C}}{\partial \bar{t}}. \quad (49)$$

The boundary and initial conditions are:

$$\bar{w}_{\beta C}(1,\theta,\bar{t})=0, \quad \bar{w}_{\beta C}(\bar{r},0,\bar{t})=1, \quad \bar{w}_{\beta C}(\bar{r},\pi,\bar{t})=1, \quad (50)$$

$$\bar{w}_{\beta C}(\bar{r},\theta,0)=0, \quad (51)$$

where

$$\bar{M}_i=\frac{M_i}{\mu}, \quad \bar{\rho}_\beta=\frac{\rho_\beta}{\rho}, \quad \bar{\alpha}=\frac{\alpha h^2}{\mu}, \quad \bar{w}_{\beta C}=\frac{w_{\beta C}}{U}, \quad \bar{r}=\frac{r}{R}, \quad \bar{t}=\frac{\mu t}{\rho R^2}. \quad (52)$$

We first transform the problem into a problem with homogeneous boundary conditions. This can be achieved by decomposing $w_{\beta C}(r,\theta,t)$ into the steady-state Couette velocity profile $w_{\beta SC}(r,\theta)$, which are expected to prevail at large times and the transient component $g_\beta(r,\theta,t)$:

$$w_{\beta C}(r,\theta,t)=w_{\beta SC}(r,\theta)-g_\beta(r,\theta,t) \quad (53)$$

The transient components satisfy the following partial differential equations

$$M_1\left(\frac{\partial^2 g_1}{\partial r^2}+\frac{1}{r}\frac{\partial g_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 g_1}{\partial\theta^2}\right)+M_2\left(\frac{\partial^2 g_2}{\partial r^2}+\frac{1}{r}\frac{\partial g_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 g_2}{\partial\theta^2}\right)$$
$$-\alpha(g_1-g_2)=\rho_1\frac{\partial g_1}{\partial t}, \quad (54)$$

$$M_3\left(\frac{\partial^2 g_1}{\partial r^2}+\frac{1}{r}\frac{\partial g_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 g_1}{\partial\theta^2}\right)+M_4\left(\frac{\partial^2 g_2}{\partial r^2}+\frac{1}{r}\frac{\partial g_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 g_2}{\partial\theta^2}\right)$$
$$+\alpha(g_1-g_2)=\rho_2\frac{\partial g_2}{\partial t} \quad (55)$$

that are consistent with the boundary and initial conditions

$$g_\beta(1,\theta,t)=0, \quad g_\beta(r,0,t)=0, \quad g_\beta(r,\pi,t)=0, \quad (56)$$

$$g_\beta(r,\theta,0)=w_{\beta SC}(r,\theta). \quad (57)$$

Finite Fourier sine and Hankel transforms are used to solve the simultaneous partial differential equations of Equations 54 and 55 with the boundary and initial conditions of Equations 56 and 57. Let $\tilde{g}_\beta(r,l,t)$ be the finite Fourier sine transform of $g_\beta(r,\theta,t)$. Transforming Equations 54 and 55 and using the initial and boundary conditions, we obtain

$$M_1\left(\frac{\partial^2\tilde{g}_1(r,l,t)}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{g}_1(r,l,t)}{\partial r}-\frac{l^2}{r^2}\tilde{g}_1(r,l,t)\right)+M_2\left(\frac{\partial^2\tilde{g}_2(r,l,t)}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{g}_2(r,l,t)}{\partial r}-\frac{l^2}{r^2}\tilde{g}_2(r,l,t)\right)$$
$$-\alpha[\tilde{g}_1(r,l,t)-\tilde{g}_2(r,l,t)]=\rho_1\frac{\partial\tilde{g}_1(r,l,t)}{\partial t}, \quad (58)$$

$$M_3\left(\frac{\partial^2\tilde{g}_1(r,l,t)}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{g}_1(r,l,t)}{\partial r}-\frac{l^2}{r^2}\tilde{g}_1(r,l,t)\right)+M_4\left(\frac{\partial^2\tilde{g}_2(r,l,t)}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{g}_2(r,l,t)}{\partial r}-\frac{l^2}{r^2}\tilde{g}_2(r,l,t)\right)$$
$$+\alpha[\tilde{g}_1(r,l,t)-\tilde{g}_2(r,l,t)]=\rho_2\frac{\partial\tilde{g}_2(r,l,t)}{\partial t}, \quad (59)$$

$$\tilde{g}_\beta(1,l,t)=0; \quad l=1,2,3,... \quad (60)$$

$$\tilde{g}_\beta(r,l,0)=\frac{[1-(-1)^l]}{l}(1-r^l). \quad (61)$$

Finite Hankel transform will be used to solve the differential system. The finite Hankel transform of order ν of a function $f(r)$, defined on $0<r<1$, is (Debnath, 1995):

$$H_\nu\{f(r);r\to\xi_m\}=\hat{f}(\xi_m)=\int_0^1 r\,f(r)J_\nu(\xi_m r)\,dr. \quad (62)$$

The associated inverse transform is

$$H_\nu^{-1}\{\hat{f}(\xi_m);\xi_m\to r\}=f(r)=2\sum_{m=1}^{\infty}\hat{f}(\xi_m)\frac{J_\nu(\xi_m r)}{J_{\nu+1}^2(\xi_m)} \quad (63)$$

where J_ν is the Bessel function of the first kind of order ν, and the summation is taken over all the positive roots $\xi_1,\xi_2,\xi_3...$ of $J_\nu(\xi_m)=0$.

Taking the finite Hankel transform of order l of Equations 58 and 59 and employing Equations 60 result in

$$\frac{d\hat{\tilde{g}}_1(\xi_m,l,t)}{dt}+k_1\hat{\tilde{g}}_1(\xi_m,l,t)+k_2\hat{\tilde{g}}_2(\xi_m,l,t)=0, \quad (64)$$

$$\frac{d\hat{\tilde{g}}_2(\xi_m,l,t)}{dt}+k_3\hat{\tilde{g}}_1(\xi_m,l,t)+k_4\hat{\tilde{g}}_2(\xi_m,l,t)=0 \quad (65)$$

where

$$k_1=\frac{M_1\xi_m^2+\alpha}{\rho_1}, \quad k_2=\frac{M_2\xi_m^2-\alpha}{\rho_1}, \quad k_3=\frac{M_3\xi_m^2-\alpha}{\rho_2}, \quad k_4=\frac{M_4\xi_m^2+\alpha}{\rho_2} \quad (66)$$

subject to the transform of Equation 61

$$\hat{\tilde{g}}_\beta(\xi_m,l,0)=\frac{[1-(-1)^l]}{l}\frac{\xi_m^l\,2^{-l}}{(l+2)\Gamma(l+1)}\,\mathrm{F}\left[\frac{2+l}{2},\frac{4+l}{2},l+1;\frac{-\xi_m^2}{4}\right]$$
$$-\frac{[1-(-1)^l]}{l}\frac{J_{l+1}(\xi_m)}{\xi_m} \quad (67)$$

where $\Gamma(z)$ is the gamma function and $\mathrm{F}(a,b,c;x)$ is the hyper-geometric function. To obtain Equation 67, we applied the following formula (Polyanin and Manzhirov, 2007)

$$\int_0^x r^{\lambda} J_{\nu}(r)dr = \frac{x^{\lambda+\nu+1}}{2^{\nu}(\lambda+\nu+1)\Gamma(\nu+1)} \mathrm{F}\left(\frac{\lambda+\nu+1}{2}, \frac{\lambda+\nu+3}{2}, \nu+1; \frac{-x^2}{4}\right). \quad (68)$$

The solution of this transformed problem is

$$\hat{\hat{g}}_1(\xi_m,l,t) = \frac{e^{-\beta_1 t}}{\varepsilon}\left\{\sinh(\varepsilon t)\left[\hat{\hat{g}}_1(\xi_m,l,0)(\beta_1-k_1) - k_2\,\hat{\hat{g}}_2(\xi_m,l,0)\right] + \varepsilon\cosh(\varepsilon t)\hat{\hat{g}}_1(\xi_m,l,0)\right\} \quad (69)$$

$$\hat{\hat{g}}_2(\xi_m,l,t) = \frac{e^{-\beta_1 t}}{k_2\varepsilon}\left\{\sinh(\varepsilon t)\left[\hat{\hat{g}}_1(\xi_m,l,0)\left(\beta_2 + k_1^2 - 2\beta_1 k_1\right) + k_2\hat{\hat{g}}_2(\xi_m,l,0)(k_1-\beta_1)\right]\right.$$
$$\left. + \varepsilon k_2 \cosh(\varepsilon t)\hat{\hat{g}}_2(\xi_m,l,0)\right\} \quad (70)$$

where

$$\beta_1 = \frac{k_1+k_4}{2}, \quad \beta_2 = k_1 k_4 - k_2 k_3, \quad \varepsilon = \sqrt{\beta_1{}^2 - \beta_2}\,. \quad (71)$$

The inverse finite Hankel transform yields

$$H_l^{-1}\{\hat{\hat{g}}_{\beta}(\xi_m,l,t);\xi_m \to r\} = \tilde{g}_{\beta}(r,l,t) = 2\sum_{m=1}^{\infty}\hat{\hat{g}}_{\beta}(\xi_m,l,t)\frac{J_l(\xi_m r)}{J_{l+1}^2(\xi_m)}\,. \quad (72)$$

Thus, the inverse finite Fourier sine transform gives the final solution as

$$F^{-1}\{\tilde{g}_{\beta}(r,l,t); l\to\theta\} = g_{\beta}(r,\theta,t) = \frac{4}{\pi}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\hat{\hat{g}}_{\beta}(\xi_m,l,t)\frac{J_l(\xi_m r)}{J_{l+1}^2(\xi_m)}\sin(l\theta)\,\cdot \quad (73)$$

Substituting Equations 45 and 73 into Equation 53, we obtain the following solution for $w_{\beta C}(r,\theta,t)$

$$w_{\beta C}(r,\theta,t) = 1 - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{[1-(-1)^k]}{k}r^k\sin(k\theta) - \frac{4}{\pi}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\hat{\hat{g}}_{\beta}(\xi_m,l,t)\frac{J_l(\xi_m r)}{J_{l+1}^2(\xi_m)}\sin(l\theta) \quad (74)$$

where the summation with the index m is taken over all the positive zeros $\xi_1, \xi_2, ...$ of the Bessel function $J_l(\xi_m)$.

STEADY POISEUILLE FLOW IN A SEMICIRCULAR DUCT

In this section, we study the steady flow of the binary mixture under consideration in a duct of semicircular cross-section. The flow is driven by externally imposed pressure gradient in the z-direction, namely $-dp/dz > 0$.

We seek a solution, compatible with mass balance equations, of the form

$$\mathbf{v}^{(\beta)} = \left\{0, 0, w_{\beta SP}(r,\theta)\right\}, \quad \rho_1 = \rho_1(r,\theta), \quad \rho_2 = \rho_2(r,\theta) \quad (75)$$

As previously stated, it is proved that the total density and the material coefficients become constants. Consequently, the equations of motion in the z-direction reduce to

$$\bar{M}_1\left(\frac{\partial^2\bar{w}_{1SP}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{w}_{1SP}}{\partial\bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2\bar{w}_{1SP}}{\partial\theta^2}\right) + \bar{M}_2\left(\frac{\partial^2\bar{w}_{2SP}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{w}_{2SP}}{\partial\bar{r}} + \frac{\partial^2\bar{w}_{2SP}}{\partial\theta^2}\right)$$
$$-\bar{\alpha}(\bar{w}_{1SP} - \bar{w}_{2SP}) = -\phi_1 \quad (76)$$

$$\bar{M}_3\left(\frac{\partial^2\bar{w}_{1SP}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{w}_{1SP}}{\partial\bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2\bar{w}_{1SP}}{\partial\theta^2}\right) + \bar{M}_4\left(\frac{\partial^2\bar{w}_{2SP}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{w}_{2SP}}{\partial\bar{r}} + \frac{\partial^2\bar{w}_{2SP}}{\partial\theta^2}\right)$$
$$+\bar{\alpha}(\bar{w}_{1SP} - \bar{w}_{2SP}) = (\phi_1 - 1) \quad (77)$$

where

$$\bar{M}_i = \frac{M_i}{\mu}, \quad \bar{\alpha} = \frac{\alpha R^2}{\mu}, \quad \bar{w}_{\beta SP} = \frac{w_{\beta SP}\,\mu}{[-dp/dz]R^2}, \quad \bar{r} = \frac{r}{R} \quad (78)$$

The adherence boundary conditions of the problem are as follows:

$$\bar{w}_{\beta SP}(1,\theta) = 0, \quad \bar{w}_{\beta SP}(\bar{r},0) = 0, \quad \bar{w}_{\beta SP}(\bar{r},\pi) = 0 \quad (79)$$

Application of the finite Fourier sine transform to Equations 76 and 77 with respect to θ, taking Equations $79_{2,3}$ into account, gives

$$M_1\left(\frac{d^2\tilde{w}_{1SP}(r,k)}{dr^2} + \frac{1}{r}\frac{d\tilde{w}_{1SP}(r,k)}{dr} - \frac{k^2}{r^2}\tilde{w}_{1SP}(r,k)\right) + M_2\left(\frac{d^2\tilde{w}_{2SP}(r,k)}{dr^2} + \frac{1}{r}\frac{d\tilde{w}_{2SP}(r,k)}{dr} - \frac{k^2}{r^2}\tilde{w}_{2SP}(r,k)\right)$$
$$-\alpha[\tilde{w}_{1SP}(r,k) - \tilde{w}_{2SP}(r,k)] = -\frac{[1-(-1)^k]}{k}\phi_1 \quad (80)$$

$$M_3\left(\frac{d^2\tilde{w}_{1SP}(r,k)}{dr^2} + \frac{1}{r}\frac{d\tilde{w}_{1SP}(r,k)}{dr} - \frac{k^2}{r^2}\tilde{w}_{1SP}(r,k)\right) + M_4\left(\frac{d^2\tilde{w}_{2SP}(r,k)}{dr^2} + \frac{1}{r}\frac{d\tilde{w}_{2SP}(r,k)}{dr} - \frac{k^2}{r^2}\tilde{w}_{2SP}(r,k)\right)$$
$$+\alpha[\tilde{w}_{1SP}(r,k) - \tilde{w}_{2SP}(r,k)] = \frac{[1-(-1)^k]}{k}(\phi_1 - 1) \quad (81)$$

subject to the transform of Equation 79_1

$$\tilde{w}_{\beta SP}(1,k) = 0 \quad (82)$$

Taking the finite Hankel transform of order k of Equations 80 and 81, and employing Equation 82 result in

$$(M_1\xi_n^2 + \alpha)\hat{\tilde{w}}_{1SP}(\xi_n,k) + (M_2\xi_n^2 - \alpha)\hat{\tilde{w}}_{2SP}(\xi_n,k)$$
$$= \frac{[1-(-1)^k]}{k}\phi_1\,\frac{\xi_n^k\,2^{-k}}{(k+2)\Gamma(k+1)}\mathrm{F}\left[\frac{2+k}{2}, \frac{4+k}{2}, k+1; \frac{-\xi_n^2}{4}\right] \quad (83)$$

$$(M_3\,\xi_n^2 - \alpha)\hat{\tilde{w}}_{1SP}(\xi_n,k) + (M_4\,\xi_n^2 + \alpha)\hat{\tilde{w}}_{2SP}(\xi_n,k)$$
$$= \frac{[1-(-1)^k]}{k}(1-\phi_1)\frac{\xi_n^k\,2^{-k}}{(k+2)\Gamma(k+1)}\mathrm{F}\left[\frac{2+k}{2}, \frac{4+k}{2}, k+1; \frac{-\xi_n^2}{4}\right] \quad (84)$$

It follows from the equations that

$$\hat{\hat{w}}_{\beta SP}(\xi_n,k)=\frac{[1-(-1)^k]}{k}\frac{\left[\xi_n^2 m_\beta+\alpha\right]}{\xi_n^2(n_1\xi_n^2+n_2)}\frac{\xi_n^k 2^{-k}}{(k+2)\Gamma(k+1)}\mathrm{F}\left[\frac{2+k}{2},\frac{4+k}{2},k+1;\frac{-\xi_n^2}{4}\right] \quad (85)$$

where

$$m_1=\phi_1(M_2+M_4)-M_2,\quad m_2=-\phi_1(M_1+M_3)+M_1 \quad (86)$$

We now obtain the solution for the velocity of the β th fluid by going back through the various substitutions:

$$w_{\beta SP}(r,\theta)=\frac{4}{\pi}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\hat{\hat{w}}_{\beta SP}(\xi_n,k)\frac{J_k(\xi_n r)}{J_{k+1}^2(\xi_n)}\sin(k\theta) \quad (87)$$

where the summation with the index n extends over all the positive roots $\xi_1,\xi_2,...$ of $J_k(\xi_n)=0$. The most important feature of the above steady solution is that the fluids in a mixture have not the same velocities. This is a result of the presence of a non-zero pressure gradient.

UNSTEADY POISEUILLE FLOW IN A SEMICIRCULAR DUCT

Finally, we discussed the problem of unsteady flow of a binary mixture of incompressible Newtonian fluids in a duct of semicircular cross-section. Suppose that the semicircular duct is filled with a stationary mixture of two fluids. At the instant $t=0$, a constant pressure gradient in the z -direction, namely $-dp/dz$, is imposed and the fluids begin to flow.

Let us assume there exist a solution of the form

$$\mathbf{v}^{(\beta)}=\left\{0,0,w_{\beta P}(r,\theta,t)\right\},\quad \rho_1=\rho_1(r,\theta,t),\quad \rho_2=\rho_2(r,\theta,t) \quad (88)$$

As previously stated, it is verified that the total density and all of the material coefficients in Equations 19 and 20 become constants. As a result, the dimensionless governing equations are as follows:

$$\bar{M}_1\left(\frac{\partial^2\bar{w}_{1P}}{\partial\bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial\bar{w}_{1P}}{\partial\bar{r}}+\frac{1}{\bar{r}^2}\frac{\partial^2\bar{w}_{1P}}{\partial\theta^2}\right)+\bar{M}_2\left(\frac{\partial^2\bar{w}_{2P}}{\partial\bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial\bar{w}_{2P}}{\partial\bar{r}}+\frac{\partial^2\bar{w}_{2P}}{\partial\theta^2}\right)$$

$$-\bar{\alpha}(\bar{w}_{1P}-\bar{w}_{2P})=\bar{\rho}_1\frac{\partial\bar{w}_{1P}}{\partial\bar{t}}-\phi_1 \quad (89)$$

$$\bar{M}_3\left(\frac{\partial^2\bar{w}_{1P}}{\partial\bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial\bar{w}_{1P}}{\partial\bar{r}}+\frac{1}{\bar{r}^2}\frac{\partial^2\bar{w}_{1P}}{\partial\theta^2}\right)+\bar{M}_4\left(\frac{\partial^2\bar{w}_{2P}}{\partial\bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial\bar{w}_{2P}}{\partial\bar{r}}+\frac{\partial^2\bar{w}_{2P}}{\partial\theta^2}\right)$$

$$+\bar{\alpha}(\bar{w}_{1P}-\bar{w}_{2P})=\bar{\rho}_2\frac{\partial\bar{w}_{2P}}{\partial\bar{t}}+(\phi_1-1) \quad (90)$$

The boundary and initial conditions are

$$\bar{w}_{\beta P}(1,\theta,\bar{t})=0,\quad \bar{w}_{\beta P}(\bar{r},0,\bar{t})=0,\quad \bar{w}_{\beta P}(\bar{r},\pi,\bar{t})=0 \quad (91)$$

$$\bar{w}_{\beta P}(\bar{r},\theta,0)=0 \quad (92)$$

where

$$\bar{M}_i=\frac{M_i}{\mu},\ \bar{\alpha}=\frac{\alpha R^2}{\mu},\ \bar{w}_{\beta P}=\frac{w_{\beta P}\mu}{[-dp/dz]R^2},\ \bar{r}=\frac{r}{R},\ \bar{t}=\frac{\mu t}{\rho R^2},\ \bar{\rho}_\beta=\frac{\rho_\beta}{\rho} \quad (93)$$

Note that all of the boundary and initial conditions given in Equations 91 and 92 are homogeneous, yet there exist a non-trivial solution, since the partial differential equation of Equation 89 and 90 are non-homogeneous.

We attempted to find a solution of the form

$$w_{\beta P}(r,\theta,t)=w_{\beta SP}(r,\theta)-h_\beta(r,\theta,t) \quad (94)$$

The components $h_\beta(r,\theta,t)$ must satisfy the differential equations and boundary conditions which are obtained from Equations 54 to 56 by writing $h_\beta(r,\theta,t)$ in place of $g_\beta(r,\theta,t)$, but with modified initial conditions which now are:

$$h_\beta(r,\theta,0)=w_{\beta SP}(r,\theta) \quad (95)$$

The procedure for determining $h_\beta(r,\theta,t)$ is the same as that previously used, so it is not repeated here. As expected, the solution given in Equation 73 for $g_\beta(r,\theta,t)$ is also valid for $h_\beta(r,\theta,t)$ provided $\hat{\hat{g}}_\beta(\xi_m,l,0)$ is replaced by the $\hat{\hat{h}}_\beta(\xi_m,l,0)$ which is given by the following analytical expression

$$\hat{\hat{h}}_\beta(\xi_m,l,0)=\frac{[1-(-1)^l]}{l}\frac{\left[\xi_m^2 m_\beta+\alpha\right]}{\xi_m^2(n_1\xi_m^2+n_2)}\frac{\xi_m^l 2^{-l}}{(l+2)\Gamma(l+1)}\mathrm{F}\left[\frac{2+l}{2},\frac{4+l}{2},l+1;\frac{-\xi_m^2}{4}\right] \quad (96)$$

We now obtain the solution for the velocity of the β th fluid by going back through the various substitutions:

$$w_{\beta P}(r,\theta,t)=\frac{4}{\pi}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\hat{\hat{w}}_{\beta SP}(\xi_n,k)\frac{J_k(\xi_n r)}{J_{k+1}^2(\xi_n)}\sin(k\theta)-\frac{4}{\pi}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\hat{\hat{h}}_\beta(\xi_m,l,t)\frac{J_l(\xi_m r)}{J_{l+1}^2(\xi_m)}\sin(l\theta) \quad (97)$$

where the summation with the index n (and m) is taken over all the positive zeros $\xi_1,\xi_2,...$ of the Bessel function $J_k(\xi_n)$ (and $J_l(\xi_m)$).

NUMERICAL RESULTS AND DISCUSSION

In this paper, we have used the classical mixture theory and theoretically studied the steady and unsteady flows of a binary mixture composed of chemically inert incompressible Newtonian fluids in a semicircular duct. Under the very special conditions mentioned earlier, exact solutions in series form for the system of coupled partial differential equations governing the velocity fields are obtained using the finite Fourier sine and Hankel transforms. We infer from these solutions that for steady problems the presence of externally applied pressure gradient brings about the relative motion between the fluids, that is, $\mathbf{v}^{(1)}-\mathbf{v}^{(2)}\neq 0$. However, in the unsteady case, the difference in velocities is due to the presence of time-dependent terms, regardless of whether or not the pressure gradient term is present.

The analytical solutions in the present work include those corresponding to pure Newtonian fluid as a special case. This provides a useful check for us. For the sake of completeness, we also present the velocity fields for a Newtonian fluid. These can be obtained from Equations 74 and 97 by letting $\bar{M}_1 = \bar{M}_2 = \bar{M}_3 = \bar{M}_4 = 1/4$ and $\bar{\rho}_1 = \bar{\rho}_2 = \phi_1 = 1/2$, respectively, as follows:

Unsteady Couette flow in a semicircular duct

$$w_{NC} = 1 - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{[1-(-1)^k]}{k} r^k \sin(k\theta) - \frac{4}{\pi}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\frac{[1-(-1)^n]}{n}\frac{J_n(\xi_i r)}{J_{n+1}^2(\xi_i)}\sin(n\theta) \times\left\{\frac{\xi_i^n\, 2^{-n}}{(n+2)\Gamma(1+n)}\,\mathrm{F}\left[\frac{2+n}{2},\frac{4+n}{2},n+1;\frac{-\xi_i^2}{4}\right] - \frac{J_{n+1}(\xi_i)}{\xi_i}\right\}\exp(-\xi_i^2\, t) \qquad (98)$$

Unsteady Poiseuille flow in a semicircular duct

$$w_{NP} = \frac{4}{\pi}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{[1-(-1)^k]}{k\xi_n^2}\frac{\xi_n^k\, 2^{-k}}{(k+2)\Gamma(k+1)}\,\mathrm{F}\left[\frac{2+k}{2},\frac{4+k}{2},k+1;\frac{-\xi_n^2}{4}\right]\frac{J_k(\xi_n r)}{J_{k+1}^2(\xi_n)}\sin(k\theta) - \frac{4}{\pi}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{[1-(-1)^l]}{l\xi_m^2}\frac{\xi_m^l\, 2^{-l}}{(2+l)\Gamma(1+l)}\,\mathrm{F}\left[\frac{2+l}{2},\frac{4+l}{2},l+1;\frac{-\xi_m^2}{4}\right]\frac{J_l(\xi_m r)}{J_{l+1}^2(\xi_m)}\exp(-\xi_m^2\, t)\sin(l\theta)\right\} \qquad (99)$$

For numerical evaluation of the results, we need the numerical values of the material coefficients appearing in Equations 74 and 97. Determination of these coefficients is not an easy task; moreover, there does not seem to be a unique way of formulating the problem, even when using a well-defined and rational theory such as mixture theory (Massoudi, 2008). To determine the viscosity coefficients μ_1, μ_2, μ_3, μ_4 and μ, we employ the following equations (Sampaio and Williams, 1977; Al-Sharif et al., 1993):

$$\mu_1 = \phi_1^2\,\eta_1 + \phi_1(1-\phi_1)\sqrt{\eta_1\eta_2},\quad \mu_2 = (1-\phi_1)^2\,\eta_2 + \phi_1(1-\phi_1)\sqrt{\eta_1\eta_2},\quad \mu_3 = \mu_4 = \phi_1(1-\phi_1)\sqrt{\eta_1\eta_2},$$
$$\mu = \phi_1^2\,\eta_1 + (1-\phi_1)^2\,\eta_2 + 2\phi_1(1-\phi_1)\sqrt{\eta_1\eta_2} \qquad (100)$$

where η_1 and η_2 are the viscosities of the unmixed fluids. Sampaio and Williams (1977) were able to derive the above formulae by employing results obtained from the kinetic theory of fluids, in the case of $\lambda_5 = 0$. In the present work, we benefit from Equation 100 to assign the reasonable values to the dimensionless material parameters $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_3$ and $\bar{M}_4$. To this end, for a mixture composed of water and oil, we first choose the volume fraction of water $\phi_1 = 0.75$, the densities of the unmixed fluids $\rho_{10} = 1000 \quad kg/m^3$, $\rho_{20} = 700 \quad kg/m^3$ and the viscosities of the unmixed fluids $\eta_1 = 0.001 \quad N\sec/m^2$, $\eta_2 = 0.01 \quad N\sec/m^2$. Later, with the help of Equations 5, 8, $52_{1,2}$ and 100, we obtain the numerical values of the dimensionless material parameters as follows:

$$\bar{M}_1 = 0.4868,\ \bar{M}_2 = \bar{M}_3 = 0.2497,\ \bar{M}_4 = 0.5132,\ \bar{\rho}_1 = 0.8108,\ \bar{\rho}_2 = 0.1892 \qquad (101)$$

We point out that a realistic evaluation of the interaction coefficient α is indeed a very difficult task as stated previously. This coefficient should ideally be determined from experiments. However, there is currently no available systematic tabulation of α for the mixture of two fluids. To plot the velocity profiles and develop some qualitative feelings about how $\bar{\alpha}$ affects velocity distributions, we choose the values of $\bar{\alpha}$ arbitrarily. We have two objectives for the remainder of this section: (1) to discuss the reliability of the solutions in series form, and (2) to plot the velocity distributions for the fluids in the mixture. We make the calculations required to achieve these objectives by employing the values of the parameters in Equation 101.

We discussed the reliability of the series solutions of Equations 74 and 97. In practice, the convergence behavior of Equations 74 and 97 is quite reasonable for large values of time. However, these solutions can also be used for small values of time provided the number of the terms in the series expansions is enough to yield satisfactory accuracy. To illustrate, in the case of unsteady Couette flow, for all $\bar{r}$ in $[0,1]$ at $\theta = \pi/2$ and $\bar{t} = 0.2$, we can approximate the infinite sums in Equation 74 to within 10^{-4} by using the $1{,}000$th partial sum of the single series and a partial sum of the double series with l and m running from 1 to 3. Note that the converge of the single series seems to be rather slow at the end point $\bar{r} = 1$; therefore, we take account of as many as $1{,}000$ terms in this series. On the other hand, for all $\bar{r}$ in $[0,1]$ at $\theta = \pi/2$ and $\bar{t} = 0.03$, the same order of accuracy is achieved using the terms of the double series with l and m up to 14, while keeping the number of the terms in the single series fixed at $1{,}000$.

To gain an insight into the patterns of flow, a few representative velocity profiles for the fluids in the mixture and pure Newtonian fluid have been plotted as a function of the dimensionless radial distance $\bar{r}$ for different values of $\bar{\alpha}$ and $\bar{t}$, keeping the remaining parameters fixed at the values given in Equation 101. From Figures 2 to 7, we observe how the velocity profiles grow with increasing time and approach asymptotically the steady-state velocity profiles. It is clear that the assumptions of Equations 53 and 94 are convenient, since the unsteady problems studied here approach the steady solutions as $\bar{t} \to \infty$. On comparing Figure 2 with Figure 4 or Figure 5 with Figure 7, we arrive at the conclusion that with an

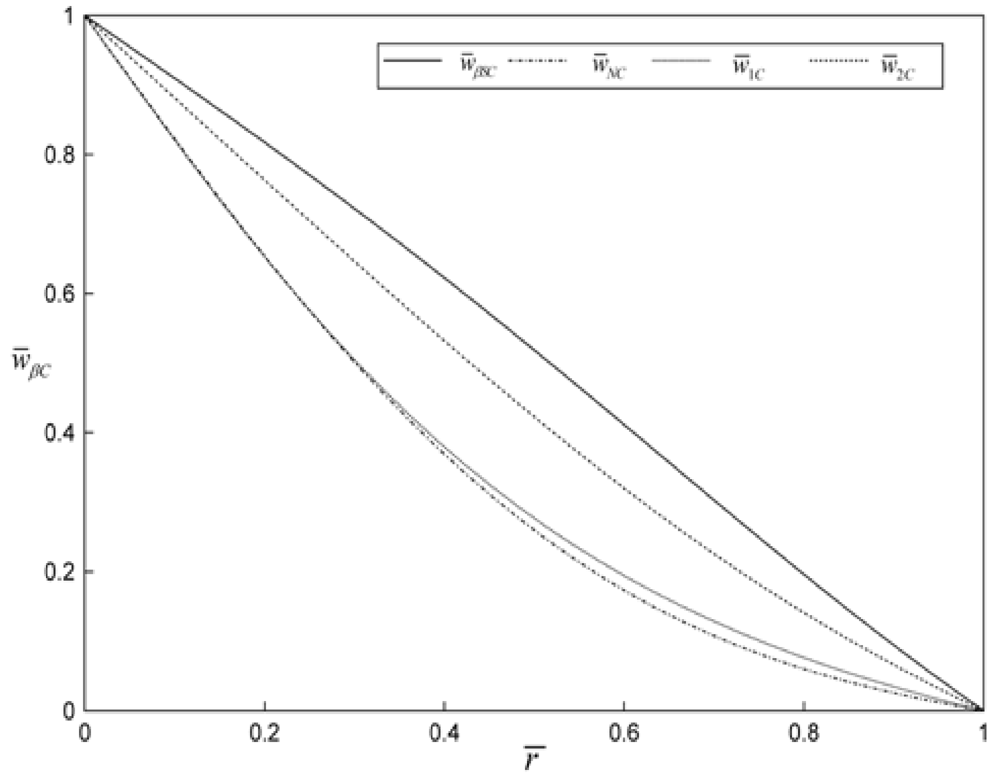

Figure 2. Velocity profiles of Couette flow in a semicircular duct for $\theta = \pi/2$, $\overline{\alpha} = 10$, $\overline{t} = 0.05$

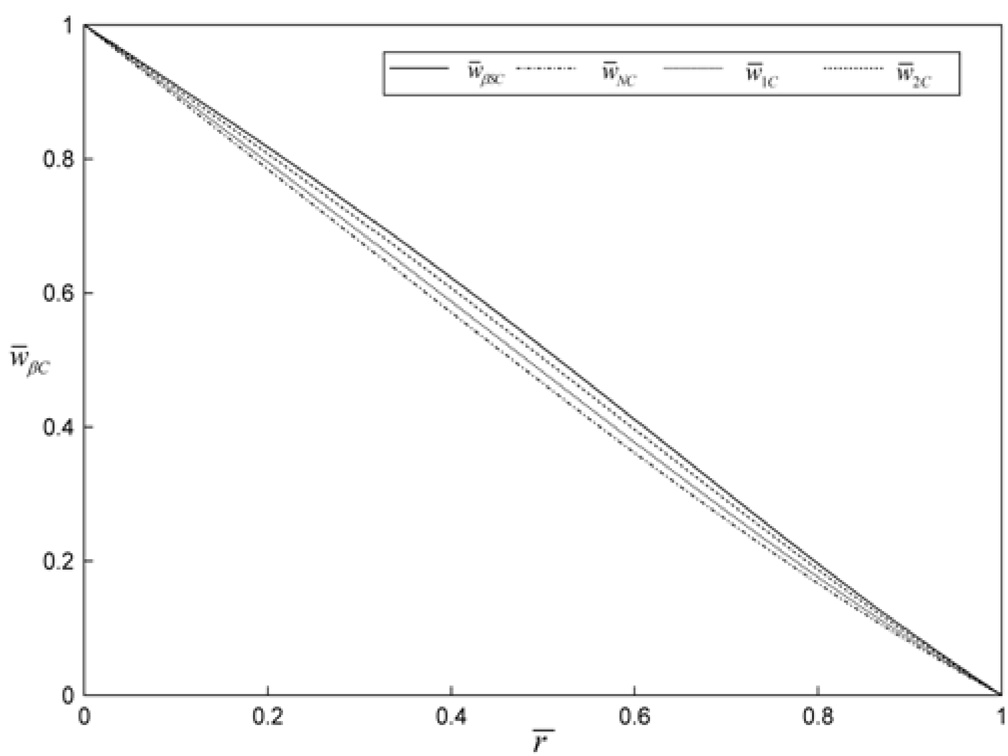

Figure 3. Velocity profiles of Couette flow in a semicircular duct for $\theta = \pi/2$, $\overline{\alpha} = 10$, $\overline{t} = 0.15$.

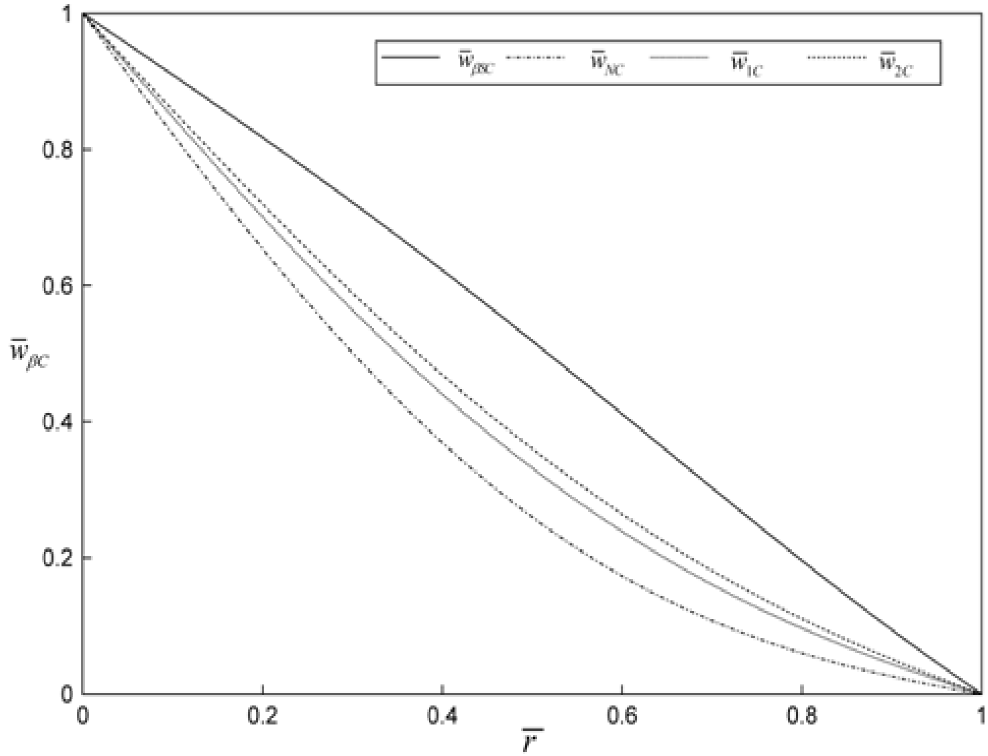

Figure 4. Velocity profiles of Couette flow in a semicircular duct for $\theta = \pi/2$, $\overline{\alpha} = 50$, $\overline{t} = 0.05$.

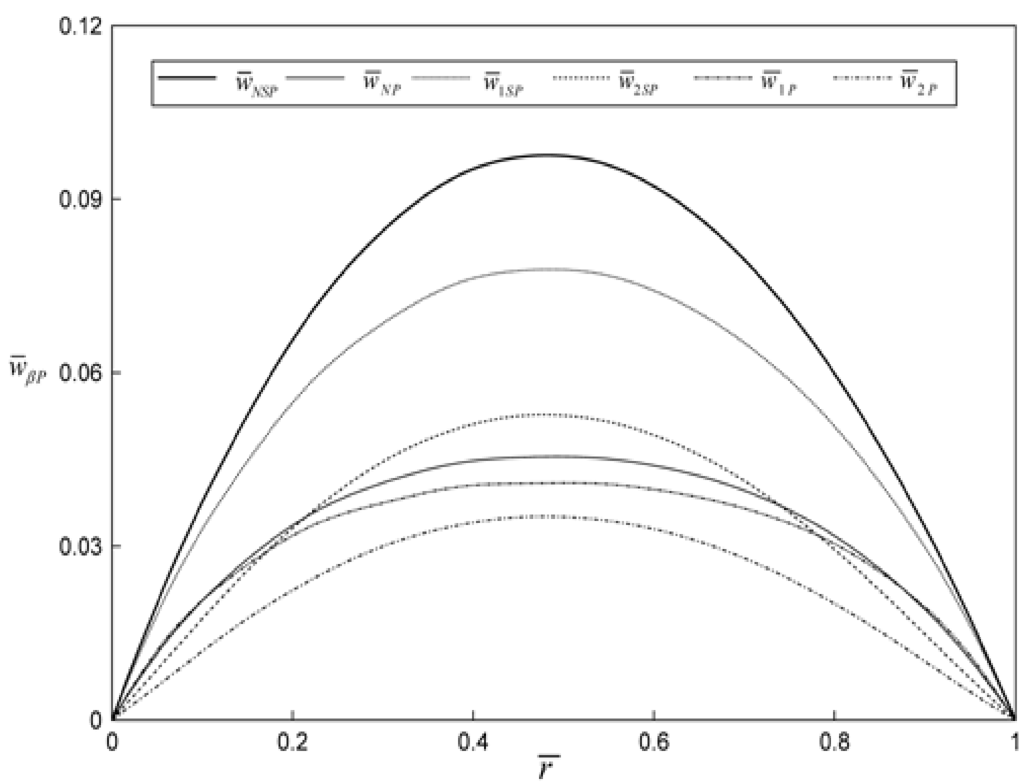

Figure 5. Velocity profiles of Poiseuille flow in a semicircular duct for $\theta = \pi/2$, $\overline{\alpha} = 10$, $\overline{t} = 0.05$.

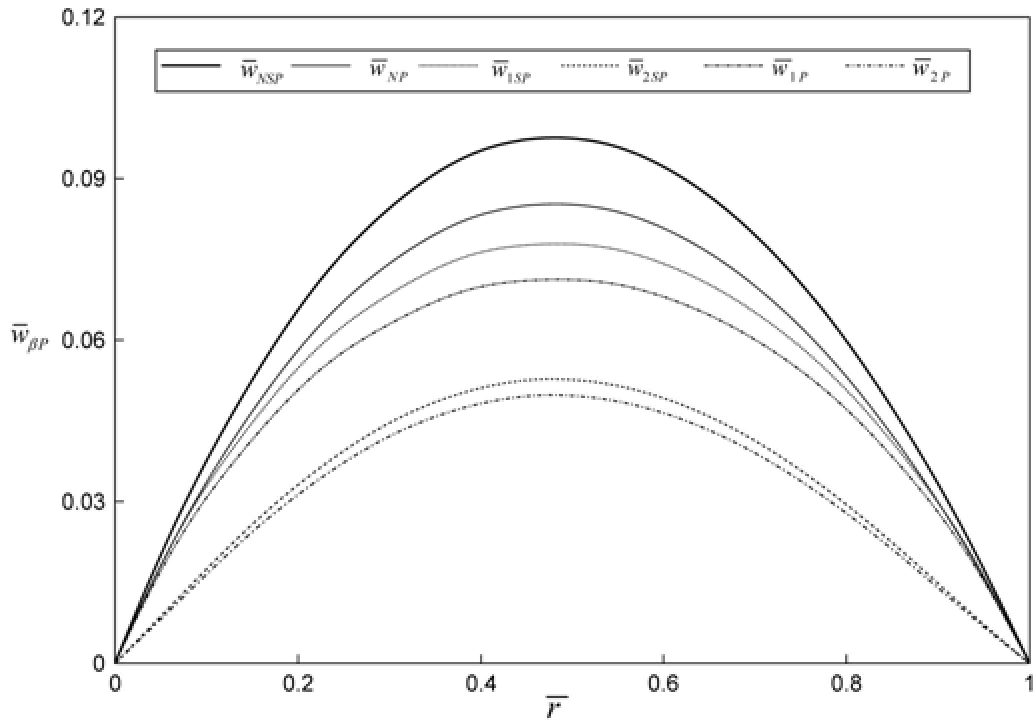

Figure 6. Velocity profiles of Poiseuille flow in a semicircular duct for $\theta = \pi/2$, $\overline{\alpha} = 10$, $\overline{t} = 0.15$.

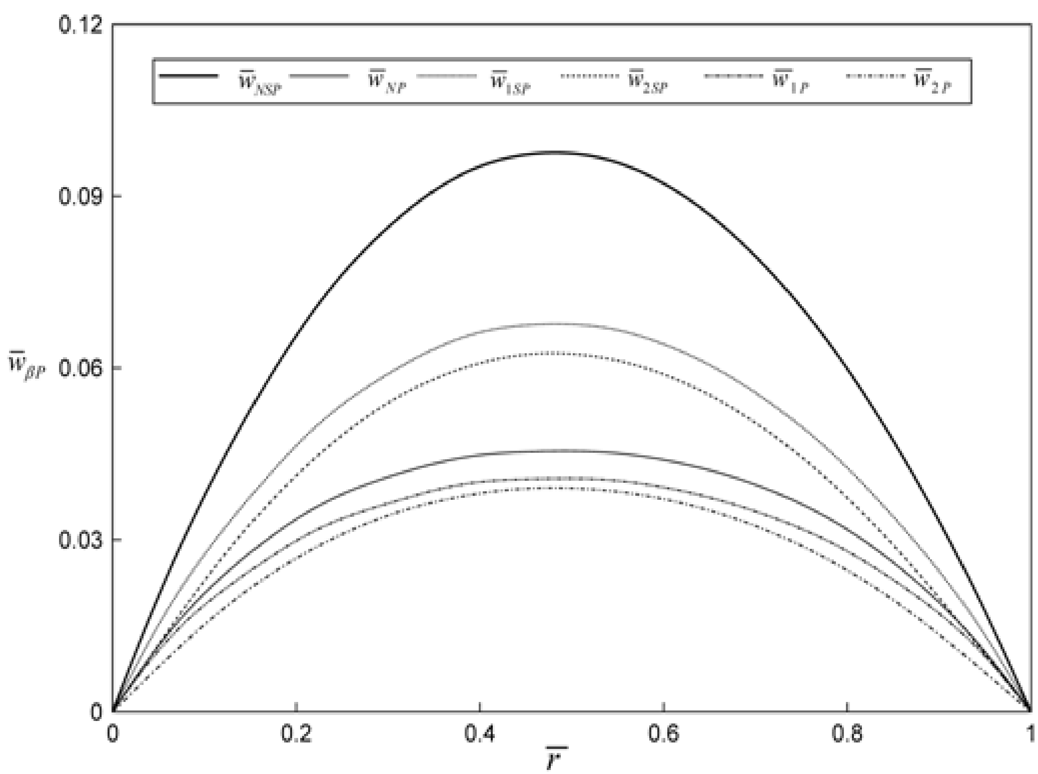

Figure 7. Velocity profiles of Poiseuille flow in a semicircular duct for $\theta = \pi/2$, $\overline{\alpha} = 50$, $\overline{t} = 0.05$.

increase in the coefficient of interaction $\bar{\alpha}$, characterized by the drag force between the two constituents, the mixture tends to behave as a single continuum. As a result, it is not difficult to predict that the fluid particles belonging to both constituents will have the same velocity at a given point in the mixture as $\bar{\alpha} \to \infty$.

The validity of the analytical solutions presented in this work can only be judged by comparing them with experimental results. Unfortunately, to the best of our knowledge no experimental data is available for direct comparison for the problem under discussion. For this reason, it is not possible to comment with any certainty on the relative merits of the constitutive equations used here. The researcher of necessity has to rely on the mixture theory to produce the correct results qualitatively at least.

Conclusions

In the present paper, we have considered the steady and unsteady flows of a binary mixture including chemically inert incompressible Newtonian fluids in a semicircular duct. Exact solutions in series form, which belong to a special semi-inverse class, for the system of coupled partial differential equations governing the velocity fields are obtained using the finite Fourier sine and Hankel integral transforms. These solutions in series form are rapidly convergent for large values of time but more slowly convergent for small values of time. If some conditions are satisfied, the series which is slowly convergent can also be used for small values of time without any difficulty. It is worth noting that for $\bar{M}_1 = \bar{M}_2 = \bar{M}_3 = \bar{M}_4 = 1/4$ and $\bar{\rho}_1 = \bar{\rho}_2 = \bar{\phi}_1 = 1/2$, Equations 74 and 97 reduce to the classical solutions of a single incompressible Newtonian fluid. This gave us confidence regarding the analytical calculations. Since experimental data on the problem under investigation are not readily available, theoretical predictions resulting from the mathematical model may be available for experimental verification.

ACKNOWLEDGEMENT

We would like to thank the editor and referee for their useful comments and suggestions regarding an earlier version of this paper.

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