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A statistical analysis of Lorentz invariance in entropy

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After relativity was introduced by Einstein in 1905, several attempts were made to reformulate the other fields of physics to make them consistent with the postulates of Relativity. However, relativistic reformulation remains a bone of contention even today after over a hundred years since Einstein's epoch making papers. This is mainly due to seemingly conflicting yet consistent ways of defining the macroscopic variables to be Lorentz invariant. Here, it will be shown from statistical origins that the current definition of entropy is untenable in relativistic scenarios.

Key words: Statistical mechanics, relativistic thermodynamics, entropy, Lorentz invariance.

INTRODUCTION

There are various ways to go about transforming the first and second laws of thermodynamics. One approach (Plank-Einstein) is to consider the invariance of pressure and obtain (Callen and Horwitz, 1971):

$$V = \frac{V_0}{\gamma}$$

From standard length contraction formulation. We are going to use standard notation implying:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

P is the pressure, U is the internal energy, Q is the Heat content, V is the volume and T is the temperature. In the γ term, v is the relative velocity of the frame while c is the speed of light. Also, the subscript $_0$ has been used to indicate the macroscopic variables in the rest frame). And therefore from standard Charles' Law:

$$T = \frac{T_0}{\gamma}$$

Also extending using the first law ($dU = dQ + PdV$):

$$Q = \frac{Q_0}{\gamma}$$

The second approach (Ott) is to consider the conservation of momentum before and after the thermodynamic process to get inverse of the above relations (Dunkel et al., 2009).

Simply by considering the energy transferred as seen from a moving frame (Rothenstein and Zaharie, 2003):

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$$Q = Q_0\gamma$$

And therefore by the invariance of entropy

$$T = T_0\gamma$$

This will also lead to:

$$V = V_0\gamma$$

Note, that since the first law has been used to obtain these relations, none of them violate it or provide any inconsistencies to the definition of entropy which is considered invariant in both the cases. The problem of course arises when we choose to consider the results of an observation. Thus here, we choose to look at a more fundamental way by looking at the statistical origins of entropy.

REFORMULATING ENTROPY

Let's begin with Boltzmann's ubiquitous equation:

$$S = k \ln W$$

There was a great deal of controversy, when this equation was introduced by Boltzmann a century ago (Campisi and Kobe, 2010). Succinctly Einstein had said, the equation lacks proper theoretical basis and the concept of microstates is heuristic at best. However, nevertheless it does simplify quite a lot of scenarios and a relativistic reformulation is much needed.

Modified H-Theorem

A most straightforward quasi-derivation comes from Boltzmann's own H-Theorem which states:

$$\frac{dH}{dt} \le 0$$

$\mathit{If}\ f_1\ \mathit{satisfies\ the}$ Boltzmann Equations, Where H(f) is defined as:

$$H[f] = \int d^3p\, d^3q\, f_1(p,q,t) \ln(p,q,t)$$

The Boltzmann equations which can easily be derived from Louiville's theorem which states:

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{force} + \left(\frac{\partial f}{\partial t}\right)_{diffusion} + \left(\frac{\partial f}{\partial t}\right)_{collision}$$

Applying the Louiville's theorem here, for phase space distributions gives us the compact form:

$$\frac{\partial f}{\partial t} + \frac{p}{m}.\nabla\mathbf{f} + \mathbf{F}.\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{collision}$$

Extending, this to make it Lorentz invariant gives us (Clemmow and Wilson, 1957):

$$\frac{1}{c\beta}\frac{\partial f}{\partial t} + u_\lambda \frac{\partial f}{\partial x_\lambda} + \frac{1}{mc^2\beta}F_\lambda \frac{\partial f}{\partial u_\lambda} + \frac{1}{mc^2\beta} f \frac{\partial F_\lambda}{\partial u_\lambda} = 0;\ for \left(\frac{\partial f}{\partial t}\right)_{collision} = 0$$

Here, $\begin{cases} u_\lambda \text{ is the three velocity} \\ F_\lambda \text{ is the three force} \\ \beta \equiv \sqrt{1 - v^2/c^2} \end{cases}$

Compared to the original non-relativistic scenario, the below equation due to Clemmow and Wilson contain the additional $\frac{1}{mc^2\beta} f \frac{\partial F_\lambda}{\partial u_\lambda}$ term which although not apparent is simply due to the first order Taylor-Expansion of the Jacobian of transformation from $f(t, x_\lambda, u_\lambda) \to f(t + \delta t, x_\lambda + \delta x_\lambda, u_\lambda + \delta u_\lambda)$. However, what is important is that this term will have a non-trivial effect on the resulting H-Theorem.

Coming back to the H-theorem, we see that the only physically useful function f which satisfies the Boltzmann equations is a probability distribution. Note, that there might be other functions which also satisfy the above criteria but are not relevant in this context. Thus, this function must be an invariant. However, engaging in a simple Jacobian of transformations from:

$$\begin{cases} q \to q' \\ p \to p' \\ t \to t' \end{cases}$$

Where the primed co-ordinates denote the relativistic scenario, we see that the integral is no longer of the same form.

Thus, as pointed out, because of the additional factor, H(f) does not retain its form and needs suitable modifications, although the moot idea remains same. It's important to note though that f is still invariant due to our above made assumption.

There is an important caveat to make here: Namely the Loschmidt paradox wherein Boltzmann supposedly obtains irreversibility using Louiville's theorem based on Newton's equations which are time symmetric. However, we will not digress and it will suffice to say here that this was resolved by assuming that initial states have low entropy which evolve to higher entropy over time.

Modified Helmholtz theorem

Campisi and Kobe (2010) demonstrated that for a Hamiltonian of a system given by (Campisi and Kobe, 2010):

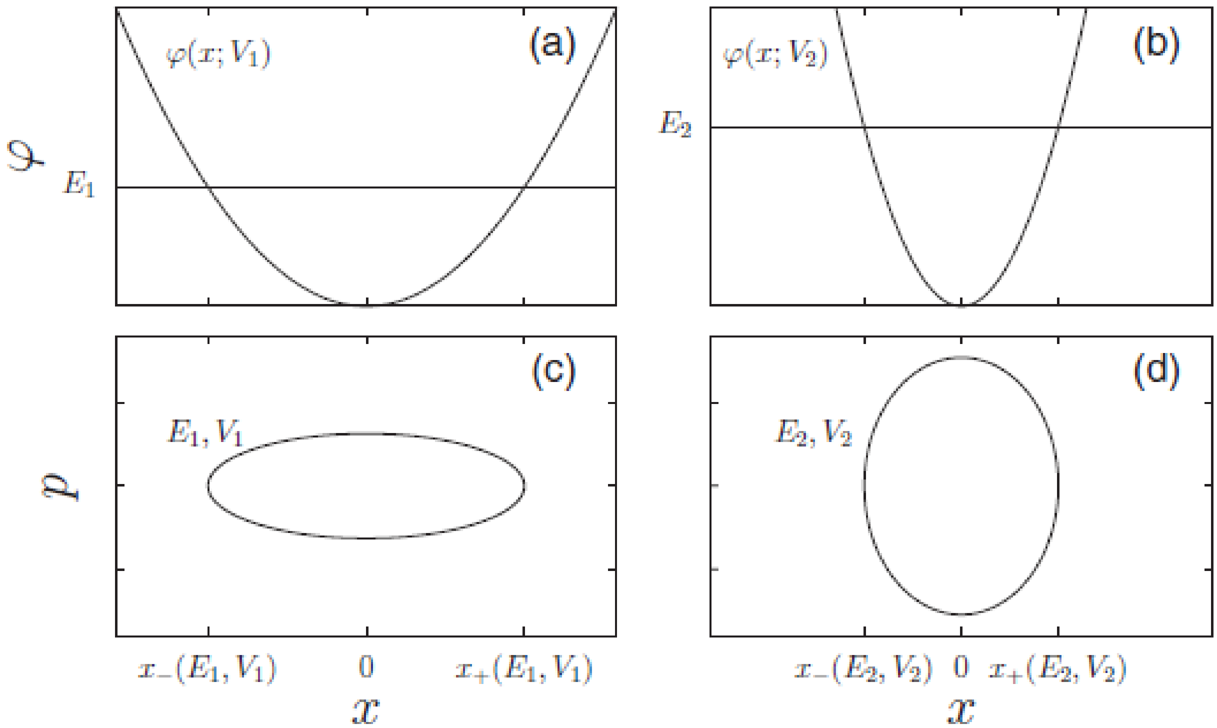

Figure 1. Point particle in the U-shaped potential $\phi(x;V)$=mV^2x^2/2. (a) Shape of the potential for a V=V_1. (b) Shape of the potential for a V=V_2. (c) Phase space orbit corresponding to the potential $\phi(x;V)$ at energy E_1. (d) Phase space orbit corresponding to the potential $\phi(x;V)$ at energy E_2. The two quantities *E,V,* uniquely determine one "state," that is, one closed orbit in phase space.

$$H(x,p;V) = K(p) + \phi(x;V)$$

Here $K(p) = \frac{p^2}{2m}$ is the kinetic energy and p is the momentum. The particle is considered to be moving in a U-shaped potential $\phi(x)$ as illustrated subsequently. Also note that V is some externally controllable parameter so that $\phi = \phi(x;V)$. Figure 1 is due to Campisi and Kobe (2010).

For a fixed V, the particle's energy E is a constant of motion. For simplicity, we define the zero energy in such a way that the minimum potential is 0, regardless of the value of V.

Once E and V are specified, the orbit of the particle in phase space is fully determined. Now applying Helmholtz theorem (Campisi and Kobe, 2010):

A function S(E,V) satisfying the following equations exist and is given by:

$$S(E,V) = k \ln 2 \int_{x_{-(E,V)}}^{x_{+(E,V)}} \sqrt{(2m(E-\phi(x;V))}dx$$

Where the equations to be satisfied are:

$$\begin{cases} \frac{\partial S(E,V)}{\partial E} = \frac{1}{T(E,V)} \\ \frac{\partial S(E,V)}{\partial V} = \frac{P(E,V)}{T(E,V)} \end{cases}$$

The entropy can be written more compactly as:

$$S(E,V) = k \ln \oint p \, dx$$

Where $p = \sqrt{(2m(E-\phi(x;V))}$ which is called the reduced action (Campisi and Kobe, 2010). It is basically the area θ in phase space enclosed by the orbit of energy E and parameter V,

$$S(E,V) = k \ln \theta(E,V)$$

Where, $\theta(E,V) = \int_{H(x,p;V)\leq E} dp\, dx.$

What is remarkable about the above result is that it suggests that there exists a consistent one dimensional mechanical counterpart of entropy given by the logarithm of the phase space volume enclosed by the curve of constant energy $H(x,p;V) = E$.

To generalize the model to more degrees of freedom, we find:

$$< f >_t = \frac{2m}{\tau} \int_{x_-}^{x_+} \frac{dx}{p(x)} f(x, p(x)); \ using \ dt = \frac{mdx}{p(x)}$$

Also using:

$$\int dp \, \delta \left(\frac{p^2}{2m} + \phi(x; V) - E \right) = \frac{2m}{p(x)}$$

Further, this allows us to define the normalized space probability density function as:

$$\rho(x, p; E, V) = \frac{1}{\tau(E, V)} \delta \left(\frac{p^2}{2m} + \phi(x; V) - E \right)$$

Thus to make a meaningful formulation, we extend to N-particle system in 3 dimensions with 3N degrees of freedom.

Thus the Hamiltonian for a N-particle system is (Campisi and Kobe, 2010):

$$H_N(\boldsymbol{q}, \boldsymbol{p}; V) = K_N(\boldsymbol{p}) + \phi_N(\boldsymbol{q}; V)$$

Thus in further analogy, we define the micro canonical probability distribution as:

$$\rho_N(\boldsymbol{q}, \boldsymbol{p}; E, V) = \frac{1}{\Omega_N(E, V)} \delta(E - H_N(\boldsymbol{q}, \boldsymbol{p}; V))$$

Where the normalization, Ω_N is given by:

$$\Omega_N(E, V) = \int \dots \int \delta(E - H_N(\boldsymbol{q}, \boldsymbol{p}; V)) d\boldsymbol{q} \, d\boldsymbol{p}$$

Generalized Helmholtz theorem

From the above we obtain the generalised Helmholtz theorem (Campisi and Kobe, 2010):

$$S_N(E, V) = k \ln \phi_N(E, V)$$

Where, $\phi_N(E, V) = \int \dots \int_{H(x, p; V) \leq E} d\boldsymbol{p} \, d\boldsymbol{x}$

Boltzmann principle

For a system composed of a large number of particles that interact through short range forces, the phase space volume approaches $\phi_N(E) \propto e^E$. Because $\Omega_N = \frac{\partial \phi_N}{\partial E}$, we also have $\phi_N \propto \Omega_N$ (From above definition of normalization).

Since Ω_N is a measure of the shell constant energy $H_N(\boldsymbol{q}, \boldsymbol{p}; V) = E$, it is proportional to the number of microstates (W) (Campisi and Kobe, 2010).

This therefore proves Boltzmann's equation to an arbitrary constant which is irrelevant anyway.

Relativistic formulation

Now, for the adiabatic scenario, $E^2 - p^2 c^2$ is an invariant and conserved quantity. Replacing back in:

$$S(E, V) = k \ln \oint p \, dx$$

We get,

$$S(E, V) = k' \ln \oint \sqrt{E^2 - m_0^2 c^4} \, dx$$

Here, k' is the modified constant term after dividing by the factor proportional to the speed of light term.

If we stick to the above formulation, since the energy doesn't remain constant, the orbit of energy E in phase space varies leading to an unbounded orbit potentially leading to an untenable definition for entropy. Even more important is that the phase space volume adds another exponential factor to the above, giving us a different definition for Boltzmann's eponymous equation. However, all of the above is based on our tacit assumption that entropy should be a relativistic invariant. Moreover, what is important is that to note is we find that simply making a relativistic reformulation without changing our definition makes Boltzmann's definition of entropy void in high velocity domains.

CONCLUSION

The above text shows that the current definition of entropy is untenable in the relativistic limit and needs to be suitably modified. Further, from above the distributions have also been shown to be derivable for high speed limits from a statistical perspective.

Conflict of Interest

The authors have not declared any conflict of interest.

REFERENCES

Callen H, Horwitz G (1971). Relativistic thermodynamics. Am. J. Phys. 39:938.

Campisi M, Kobe D (2010). Derivation of Boltzmann principle. Am. J. Phys. 78:608.

Clemmow PC, Wilson AJ (1957). A relativistic form of Boltzmann's Equations in the Absence of Collisions. Math. Proc. Camb. Philos.

Soc. 53:220.
Dunkel J, Hangi P, Hilbert S (2009). Non local observables and lightcone-averaging in relativistic thermodynamics. Nat. Phys. 5:741.
Rothenstein B, Zaharie I (2003). Relativistic thermodynamics for the introductory physics course. J. Theoretics. 9:2.