academicJournals

Vol. 4(8), pp. 103-115, December 2013
DOI: 10.5897/JCEMS11.036
ISSN 2141-6605 © 2013 Academic Journals
http://www.academicjournals.org/JCEMS

Journal of Chemical Engineering and Materials Science

Full Length Research Paper

Time dependent mathematical model of air pollutants emitted from time-dependent elevated line source into a stable atmospheric boundary layer

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Accepted 30 November, 2011

A time dependent atmospheric model is represented for chemically reactive primary pollutants emitted from an elevated line source into a stable atmospheric boundary layer, over rough surface terrains. The time dependent model was obtained through an analytical solution from a multiple Inverse Laplace transform of the atmospheric diffusion equation with the parabolic eddy-diffusion coefficient (exchange coefficient) and the wind velocity as functions of vertical height. The pollutants were considered to be of chemically reactive primary pollutants emitted from a time-dependent line source of, step-Function type. In order to facilitate the application of the model, the results for the general situation that includes chemical reaction rate and time dependent line source are incorporated in the model. In this model, the effect of step-function type elevated line source studied near the source. The results obtained in this model are compared with that of the continuous line sources in the previous work. The results obtained in this model are of good agreement with that of the continuous line source.

Key words: Stable-atmospheric boundary layer (SBL), Step-function type line sources, parabolic eddy-diffusion coefficient.

INTRODUCTION

The study of the atmospheric dispersion of air pollutants from different types of sources received a great deal of attention during the last few decades. The effects of various sources of chemically reactive pollutants are been investigated in the recent studies. In addition, the depletion of pollutants plumes will effect the pollutants concentration especially when the deposition occurs over a long distance (Robson, 1983). An analytical model for air pollutant transport and dispersion from a point source is studied (Donald, 1976). This paper addresses an analytical model of air pollutant dispersion and removal from a step-function type elevated line sources. The removal (washed out) of pollutant depends on the nature of gaseous effluents emitted from the sources and the percentage of water vapors present in the atmosphere. The step-function type of line sources were assumed the series of industries or highways on during certain period of time and not continuous. The results were validated by comparing that of Robson (1987) by chemically non-reactive and time independent case. It is interesting to note that when there is no chemical reaction of primary pollutants, the results of the present model coincide with the results of the model presented by Robson (1987). This paper analyzed unsteady state dispersion of pollutants from a line source into stable atmosphere boundary layer. The quadratic diffusion coefficient and step-function type line source with variable wind profile like constant, constant shear and parabolic wind are

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incorporated. We adopted multiple Laplace transform and Green's Function technique for the effective solution. In most of the previous models, numerical methods used either fully or partly. In model where the Laplace transform is generally inverted numerically (Robson, 1987). Whereas in this model, the multiple Laplace transform inverted analytically this gives an exact solution. The solutions of this model verified with that of (Robson, 1987) by assuming no delayed removal (chemical reaction rate) and time independent case.

METHODOLOGY

We consider the whole atmospheric boundary layer as subdivided into two layers viz: (i) the surface layer (SL) that is, $0 \le z \le z_o$ below the source height and (ii) the layer above SL and up to the inversion layer that is, protected zone $z_o \le z \le H$ above source height. Here, H is the height of SL and z_o is the source height. The situation analyzed is that the chemically reactive pollutant was emitted from an elevated line source, which is of step-function type. It is assumed that the emission of contaminant are in the gaseous state from a time dependent elevated line source of infinite length into a stable ABL, the variable wind velocity depending on the nature of the surface terrains viz. flat, buildup areas and mountains like elevated regions. The diffusion coefficient (exchange coefficient) was assumed to be of quadratic in nature, the boundary layer assumed to be of stable atmospheric boundary layer, where the mixing of pollutants is limited. To study the variation in the effect of pollutants, the various wind profiles like constant, constant shear and parabolic type are considered. The present work focused on the step-function type of source, where certain industries work during the certain allotted period and then switched off. The pollutant transport governed by the atmospheric advection-diffusion equation with delayed removal is as follows:

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=\frac{\partial}{\partial x}\left(K_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right)-K''C+S \tag{1}$$

Where, C is the concentration of pollutants in the atmosphere at any location (x, y, z) at time t; u, v, w are the components of velocity; K_x, K_y, K_z are the coefficients of eddy diffusivity (exchange coefficient) in the x, y, z directions respectively; K'' is the first order delayed removal rate, and S is the source of the effluents. The height of the atmospheric boundary layer is assumed to be z = H, from the surface.

The basic Equation (1) is simplified by assuming the following assumptions:

1. Model is of time dependent and advection dominates over horizontal diffusion

$$u\frac{\partial \mathrm{C}}{\partial \mathrm{x}} >> \frac{\partial}{\partial \mathrm{x}}\left(K_x\frac{\partial \mathrm{C}}{\partial \mathrm{x}}\right)$$

2. The line source is in the direction of y-axis, which leads the concentration gradient and flux gradient becomes zero along y-direction.

$$v\frac{\partial \mathrm{C}}{\partial \mathrm{y}}=0 \quad \& \quad \frac{\partial}{\partial \mathrm{y}}\left(K_y\frac{\partial \mathrm{C}}{\partial \mathrm{y}}\right)=0$$

3. The vertical diffusion dominates over advection

$$\frac{\partial}{\partial \mathrm{z}}\left(K_z\frac{\partial \mathrm{C}}{\partial \mathrm{z}}\right) >> w\frac{\partial C}{\partial \mathrm{z}}$$

4. The source (S) is of time dependent line source and is assumed to be

$$S = QW(t)\delta(x-x_0)\delta(z-z_0)$$

5. The diffusion coefficient (exchange coefficient) is of parabolic form $(1-z)^2$, suited for stable atmospheric boundary layer, where the penetration of pollutants through the inversion layer does not takes place.

Based on the aforementioned assumptions, the transport and diffusion Equation (1) becomes:

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}=\frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right)-K''C+QW(t)\delta(x-x_0)\delta(z-z_0) \tag{2}$$

Initial and boundary conditions are:

$$C(x,y,z)=0\,, \quad at\; t=0, \qquad \forall\; x, z \in R^+ \tag{3a}$$

$$C(x,y,z)=0\,, \quad at\; x=0, \qquad t \ge 0, \quad z \ge 0 \tag{3b}$$

$$K_z\frac{\partial C(x,y,z)}{\partial z}=0, \quad z=1 \qquad \forall \quad x,t \ge 0 \tag{3c}$$

$$K_z\frac{\partial C(x,y,z)}{\partial z}=0, \quad z=0 \qquad \forall \quad x,t \ge 0 \tag{3d}$$

$$W(t) = w_o U(t-t_o) = \begin{cases} w_o\,, & 0 < t \le t_o \\ 0\,, & t > t_o \end{cases} \text{------ step-function type.} \tag{4}$$

Introducing the non-dimensional parameters

$$Z=\frac{z}{H},\quad \frac{K_z}{K_o}=(1-Z)^2,\; X=\frac{K_o}{H^2u_o}x,\quad \frac{u(z)}{u_o}=\left[\frac{z}{H}\right]^n=Z^n,\quad V_D=\frac{K_o}{H^2}v_d$$

$$\mathrm{T}=\frac{\mathrm{K_o}}{\mathrm{H}^2}\mathrm{t},\ \mathrm{C}=\frac{\mathrm{H}<\mathrm{u}>}{\mathrm{W_oQ}}\mathrm{c}, \quad \text{w here} \quad <\mathrm{u}>=\frac{\mathrm{u_o}}{\mathrm{n}+1},\quad \mathrm{W(T)}=\frac{\mathrm{W(t)}}{\mathrm{w_o}},$$

And $u_o = \dfrac{4.7u_* H}{kL}, K_o = \dfrac{k}{4.7} u_* L$ (Robson, 1987)

where: $u_* =$ is friction velocity; L=the Monin-Obukovlenght; k=Vankarman constant; H=the boundary layer height and n=the constant decided on the stability of boundary layer.

In addition, non-dimensional source term is as follows:

$$W(T) = \frac{W(t)}{W_o} = U(T - T_o)$$

Incorporating all the aforementioned non-dimensional parameters in the Equation (2) and dropping out the capitals, we get:

$$\frac{\partial C}{\partial t} + z^n \frac{\partial C}{\partial x} = \frac{\partial}{\partial z}\left((1-z)^2 \frac{\partial C}{\partial z}\right) - \alpha\, C + \frac{U(t-t_o)}{n+1}\delta(x-x_0)\delta(z-z_0) \qquad (5)$$

Where, $\alpha = \dfrac{K''H^2}{K_0}$ (6)

α is the non-dimensional chemical reaction.

Initial and boundary conditions are:

$$C(x,z,t) = 0\ ,\ \ at\, x = 0\ , \qquad t \geq 0, z \geq 0 \qquad (7a)$$

$$C(x,z,t) = 0\ ,\ \ at\, t = 0\ , \qquad x \geq 0, z \geq 0 \qquad (7b)$$

$$(1\text{-}z)^2 \frac{\partial C(x,z,t)}{\partial z} = 0, \qquad at\ \ z = 1 \qquad (7c)$$

$$(1\text{-}z)^2 \frac{\partial C(x,z,t)}{\partial z} = 0, \qquad at\ \ z = 0 \qquad (7d)$$

Solution

Taking Laplace transforms along t and using initial conditions, we get:

$$s\bar{C}\ (x,z,s)\text{-}C\,(x,z,0) + z^n \frac{\partial \bar{C}}{\partial x} = \frac{\partial}{\partial z}\left((1\text{-}z)^2 \frac{\partial \bar{C}}{\partial z}\right) - \alpha \bar{C} + \frac{\overline{W(s)}}{n+1}\delta(x-x_0)\delta(z-z_0) \qquad (8)$$

Initial and Boundary conditions:

$$C(x,z,t) = 0 \quad at\, t = 0\ , \qquad x \geq 0, z \geq 0 \qquad (9a)$$

$$(1\text{-}z)^2 \frac{\partial \overline{C}(x,z,s)}{\partial z} = 0, \qquad at\ \ z = 1 \qquad (9b)$$

$$(1\text{-}z)^2 \frac{\partial \overline{C}(x,z,s)}{\partial z} = 0, \qquad at\ \ z = 0 \qquad (9c)$$

Where,

$$\bar{C}\ (x,z,s) = \int_0^\infty e^{-st} C(x,z,t)dt \qquad (10)$$

Again, taking Laplace transform over (8) along x-axis, which assumed of semi-infinite region, then it becomes

$$s\hat{C}\ (p,z,s) + z^n\left(p\hat{C}\ (p.z,s)\text{-}\bar{C}(0,z,s)\right) = \frac{d}{dz}\left((1\text{-}z)^2 \frac{d\hat{C}}{dz}\right) - \alpha\hat{C} + \frac{\overline{W(s)}}{n+1}e^{-sx_o}\delta(z-z_o) \qquad (11)$$

Where

$$\hat{C}\ (p,z,s) = \int_0^\infty e^{-st}\, \overline{C}(x,z,s)dt \qquad (12)$$

Differential Equation (11) becomes

$$\frac{d}{dz}\left((1\text{-}z)^2 \frac{d\hat{C}}{dz}\right) - (\alpha + s + p\, z^n)\hat{C} \quad = \quad -\frac{\overline{W(s)}}{n+1}e^{-sx_o}\delta(z-z_0) \qquad (13)$$

boundary conditions:

$$(1\text{-}z)^2 \frac{d\hat{C}(p,z,s)}{dz} = 0, \qquad at\ \ z = 1 \qquad (14a)$$

$$(1\text{-}z)^2 \frac{d\hat{C}(p,z,s)}{dz} = 0, \qquad at\ \ z = 0 \qquad (14b)$$

Case 1

When the wind profile is of constant (n = 0)

The movements of air near the earth's surface retarded by frictional effect proportional to the surface roughness. Thus, the nature of the terrain, the location and density of trees, the location and size of lakes, rivers, hills and building produces different wind velocity gradients in the vertical direction (Warn and Walker, 1967). The wind of constant type may be experienced in case of flat open country, lakes and seas, then the Equation (13) becomes

$$\frac{d}{dz}\left((1\text{-}z)^2 \frac{d\hat{c}}{dz}\right) - (\alpha + s + p)\hat{c} \ = \text{-}\overline{W(s)}e^{-px_o}\delta(z-z_0) \qquad (15)$$

Under the boundary conditions:

$$(1\text{-}z)^2 \frac{d\hat{C}(p,z,s)}{dz} = 0, \qquad at\ \ z = 1 \qquad (16a)$$

$$(1\text{-}z)^2 \frac{d\hat{C}(p,z,s)}{dz} = 0, \qquad at\ \ z = 0 \qquad (16b)$$

The complete solution of equation 16a and b, through Green's function technique Beck et al. (1992) assumed to be of the following form:

$$\hat{C}(p,z,s) = \begin{cases} G_1(z/z_0), & z<z_0 \\ G_2(z/z_0), & z>z_0 \end{cases} \tag{17}$$

$$\hat{C}(p,z,s) = \begin{cases} -\dfrac{\overline{W(s)}\, e^{-px_0} \phi_a(z)\phi_b(z_0)}{A_1}, & z<z_0 \\ -\dfrac{\overline{W(s)}\, e^{-px_0} \phi_a(z_o)\phi_b(z)}{A_1}, & z>z_0 \end{cases} \tag{18}$$

Where, $\phi_a(z)$ and $\phi_b(z)$ are the two independent solutions of homogeneous linear ordinary differential equation,

$$\frac{d}{dz}\left((1\text{-}z)^2 \frac{d\hat{\phi}}{dz}\right) - (\alpha + s + p)\phi = 0 \tag{19}$$

Under the boundary conditions:

$$(1\text{-}z)^2 \frac{d\phi}{dz} = 0, \quad at \quad z = 1 \tag{20a}$$

$$(1\text{-}z)^2 \frac{d\phi}{dz} = 0, \quad at \quad z = 0 \tag{20b}$$

And are found to be

$$\chi_a = (1-z)^{\frac{-(\beta+1)}{2}} + R_1(1-z)^{\frac{(\beta-1)}{2}} \tag{21a}$$

$$\chi_b = (1-z)^{\frac{(\beta-1)}{2}} \tag{21b}$$

In addition, A_1 is obtained through Wronskian of

$$W[\phi_a,\phi_b]_{z=z_0} = \phi_a(z)\phi_b'(z) - \phi_a'(z)\phi_b(z) = \frac{A_1}{(1-z_o)^2} \tag{22}$$

By Wronskian A_1 found to be $A_1 = -\beta$

Where,

$$\beta = \sqrt{1 + 4(\alpha + s + p)} \tag{23}$$

and

$$R_1 = \frac{\beta+1}{\beta-1} \tag{24}$$

The arbitrary constant R_1 determined as shown in the Equation (24) by using the boundary conditions on the solution (20a). There of the solution $\varphi_a(z)$ and $\varphi_b(z)$ satisfies the b.c's., at z = 0 and z=1 respectively. Taking inverse Laplace transform using standard tables (Erdelyi et al., 1954) the solution (18) becomes:

$$C_1(x,z,t) = \delta(t-x)\frac{e^{-(\alpha+\frac{1}{4})x}}{2\sqrt{(1-z)(1-z_0)}}\left\{\frac{e^{-\frac{g_1^2}{4x}} + e^{-\frac{g_2^2}{4x}}}{\sqrt{\pi x}} + e^{\frac{-g_2+x^2}{2}} Erfc\left(\frac{g_2}{2\sqrt{x}} - \frac{\sqrt{x}}{2}\right)\right\}, \quad z<z_0 \tag{25}$$

Where,

$$g_1 = \log\left(\frac{1-z}{1-z_o}\right) \tag{26}$$

$$g_2 = \log\left(\frac{1}{(1-z)(1-z_o)}\right) \tag{27}$$

The complete solution for $z > z_o$ is obtained by interchanging z and z_o in the above solution.

Case 2

When the wind profile is of constant shear (n=1)

The wind profile of constant shear $u(z) = z$ is experienced in the metro cities of highly buildup areas. The velocity varies linearly with vertical height then differential equation (13) is reduced to the form for quadratic diffusion coefficient.

$$\frac{d}{dz}\left((1\text{-}z)^2 \frac{d\hat{C}}{dz}\right) - (\alpha + s + pz)\hat{C} = -\frac{\overline{W(s)}}{2} e^{-px_o} \delta(z - z_o) \tag{28}$$

Following boundary conditions,

$$(1\text{-}z)^2 \frac{d\hat{C}}{dz} = 0, \quad at \quad z = 1 \tag{29a}$$

$$(1\text{-}z)^2 \frac{d\hat{C}}{dz} = 0, \quad at \quad z = 0 \tag{29b}$$

The solution of Equation 29 through Green's function technique is assumed to be of the following form:

$$\hat{C}(p,z,s)=\begin{cases} -\dfrac{W(s)e^{-px_o}\psi_a(z)\psi_b(z_o)}{2A_2}, & z<z_0 \\ -\dfrac{W(s)e^{-px_o}\psi_a(z_o)\psi_b(z)}{2A_2}, & z>z_0 \end{cases} \quad (30)$$

Where A_2 is obtained through Wronskian of $\psi_a(z)$ & $\psi_b(z)$

$$W[\psi_a,\psi_b]_{z=z_o}=\psi_a\psi_b'-\psi_a'\psi_b=\frac{A_2}{(1-z_o)^2} \quad (31)$$

$\psi_a(z)$ and $\psi_b(z)$ are the two linearly independent solutions of the homogeneous differential equation:

$$\frac{d}{dz}\left((1\text{-}z)^2\frac{d\psi}{dz}\right)-(\alpha+s+pz)\psi=0 \quad (32)$$

Under the boundary conditions:

$$(1\text{-}z)^2\frac{d\psi}{dz}=0, \qquad at \quad z=1 \quad (33a)$$

$$(1\text{-}z)^2\frac{d\psi}{dz}=0, \qquad at \quad z=0 \quad (33b)$$

Let $(1-z)=\eta$

$$\psi''+\frac{2}{\eta}\psi'+\left(\frac{p}{\eta}-\frac{\alpha+p+s}{\eta^2}\right)\psi=0 \quad (34)$$

This is the Bessel's differential equation of first kind since the order ($\nu=\sqrt{1+4(\alpha+s+p)}$) of Equation 34 is assumed to be of non-integer when it is compared with the generalized Bessel's equation

$$R''+\left(\frac{1-2m}{x}-2\alpha\right)R'+\left[p^2\alpha^2x^{2p-1}+\alpha^2+\frac{\alpha(2m-1)}{x}+\frac{m^2-p^2\upsilon^2}{x^2}\right]R=0 \quad (35)$$

(Sherwood and Reed, 1939), and the solutions are found to be

$$\psi_a(z)=\frac{1}{\sqrt{\eta}}\left[J_{-\nu}(2\sqrt{p\eta})+R_2J_\nu(2\sqrt{p\eta})\right] \quad (36)$$

ν is not an integer

$$\psi_b(z)=\frac{J_\nu(2\sqrt{p\eta})}{\sqrt{\eta}} \quad (37)$$

Where,

$$\nu=\sqrt{1+4(\alpha+s+p)} \quad (38)$$

An arbitrary constant R_2 is determined by imposing the boundary condition at z = 0 on Equation (35)

$$R_2=\frac{1}{p^\nu}\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\left(\frac{\nu+1}{\nu\text{-}1}\right) \quad (39)$$

Wronskian determines A_2 with help of the above solutions we get:

$$A_2=-\frac{\sin\nu\pi}{\pi} \quad (40)$$

The solutions of Equation 32 followed by the boundary conditions, which is defined in two regions are follows:

$$\hat{C}(p,z,s)=\begin{cases} \dfrac{\pi\overline{W(s)}}{2}\dfrac{1}{\sqrt{\eta}}\left[J_{-\nu}(2\sqrt{p\eta})+R_2J_\nu(2\sqrt{p\eta})\right]\dfrac{J_\nu(2\sqrt{p\eta_o})}{\sin\nu\pi}, & z<z_0 \\ \dfrac{\pi\overline{W(s)}}{2}\dfrac{1}{\sqrt{\eta}}\left[J_{-\nu}(2\sqrt{p\eta_o})+R_2J_\nu(2\sqrt{p\eta_o})\right]\dfrac{J_\nu(2\sqrt{p\eta})}{\sin\nu\pi}, & z>z_0 \end{cases} \quad (41)$$

As limiting case, where $p\to 0$ that is, $x>>1$, for large values of x, $J_\nu(2\sqrt{p\eta})$ and $J_{-\nu}(2\sqrt{p\eta})$ are approximated (Beck et al., 1992) as follows:

$$J_\nu(2\sqrt{p\eta})\cong\frac{(\sqrt{p\eta})^\nu}{\Gamma(1+\nu)} \quad (42)$$

$$J_{-\nu}(2\sqrt{p\eta})\cong\frac{(\sqrt{p\eta})^{-\nu}}{\Gamma(1-\nu)}, \quad \nu\notin \text{integers} \quad (43)$$

Then the solution for $z<z_o$ solution is

$$\psi_1(x,z,t) \cong U(t-x-t_0)\frac{e^{-(\alpha+\frac{1}{4})x}}{2\sqrt{(1-z)(1-z_0)}}\left\{F(x)+\frac{e^{\frac{-2h_2+x}{8}}}{2}Erfc\left(\frac{h_2}{2\sqrt{x}}-\frac{\sqrt{x}}{2}\right)\right\} \quad (44)$$

Where,

$$F(x)=\frac{e^{-\frac{h_1^2}{4x}}+e^{-\frac{h_2^2}{4x}}}{2\sqrt{\pi x}} \quad (45)$$

$$h_1=\log\left(\frac{1-z}{1-z_o}\right) \quad (46)$$

$$h_2=\log\left(\frac{1}{(1-z)(1-z_o)}\right) \quad (47)$$

This is the complete solution of the Equation (32) followed by the boundary conditions 33(a)- (b) for $z<z_o$. Similarly, the solution for $z>z_o$ can be obtained by interchanging z and z_o in the above solution. However, we dealt only with the case below the source height $z<z_o$, that is, at the ground level.(z=0).

Case 3

When the wind profile is of parabolic type (n = 2)

The wind profile of parabolic type $u(z)=z^2$ (when n = 2) experienced within the study area located near the deep valley and oceans, then the differential Equation (4.3.6) reduced to the form with diffusion coefficient (exchange coefficient) is of quadratic in nature.

$$\frac{d}{dz}\left((1\text{-}z)^2\frac{d\hat{C}}{dz}\right)-(\alpha+s+pz^2)\hat{C} = -\frac{\overline{W(s)}}{3}e^{-px_o}\delta(z-z_0) \quad (48)$$

Boundary conditions:

$$(1\text{-}z)^2\frac{d\hat{C}}{dz}=0, \qquad at \quad z=1 \quad (49a)$$

$$(1\text{-}z)^2\frac{d\hat{C}}{dz}=0, \qquad at \quad z=0 \quad (49b)$$

Solutions of Equation (48) through Green's function are assumed to be of the form:

$$\hat{C}(p,z,s)=-\overline{W(s)}e^{-px_o}\begin{cases}\dfrac{\xi_a(z)\xi_b(z_o)}{3A_3}, & z<z_0\\[2ex] \dfrac{\xi_a(z_0)\xi_b(z)}{3A_3}, & z>z_0\end{cases} \quad (50)$$

Where A_3 is determined by the Wronskian

$$W[\xi_a,\xi_b]_{z=z_0}=\xi_a\xi_b{'}-\xi_a{'}\xi_b=\frac{A_3}{(1-z_o)^2} \quad (51)$$

In addition, $\xi_a(z)$ and $\xi_b(z)$ are the linearly independent solutions of the differential equation.

$$\frac{d}{dz}\left((1\text{-}z)^2\frac{d\xi}{dz}\right)-(\alpha+s_1+pz)\xi=0 \quad (52)$$

With boundary conditions

$$(1\text{-}z)^2\frac{d\xi}{dz}=0, \qquad at \quad z=1 \quad (53a)$$

$$(1\text{-}z)^2\frac{d\xi}{dz}=0, \qquad at \quad z=0 \quad (53b)$$

Taking,

$$\mu=2\sqrt{p}(1-z) \quad (54)$$

in the above differential equation then it reduces to the form

$$\mu^2\frac{d^2\xi}{d\mu^2}+2\mu\frac{d\xi}{d\mu}+\left\{\frac{-\mu^2}{4}+\lambda_2\mu-\lambda_1\right\}\xi=0 \quad (55)$$

Taking the transformation on the independent variable

$$\xi(\mu)=\frac{1}{\mu}W(\mu) \quad (56)$$

in Equation (55), we get,

$$\frac{d^2W}{d\mu^2}+\left\{\frac{-1}{4}+\frac{\lambda_2}{\mu}-\frac{\lambda_1}{\mu^2}\right\}W=0 \quad (57)$$

This is a Whittaker's differential equation, where $\lambda_1=\alpha+s+p$

and $\lambda_2 = \sqrt{p}$ the two independent solutions of Equation (52) are the Whittaker's Confluent Hyper geometric function as follows:

$$W_a(\mu) = W_{k,-m}(\mu) + R_3 W_{k,m}(\mu) \quad (58)$$

$$W_b(\mu) = W_{k,m}(\mu) \quad (59)$$

Where,

$$W_{k,m}(\mu) = \mu^{\frac{1}{2}+m} e^{-\frac{\eta}{2}} F(\frac{1}{2} - k + m, 1 + 2m, \mu) \quad (60)$$

$$W_{k,-m}(\mu) = \mu^{\frac{1}{2}-m} e^{-\frac{\eta}{2}} F(\frac{1}{2} - k - m, 1 - 2m, \mu) \quad (61)$$

and also Inter-relation between Whittaker's to Bessel's function is derived as follows (Carslaw and Jeager)

$$F(\alpha, \beta, \frac{a^2}{4t}) = e^{\frac{a^2}{4t}} (\alpha - \beta)!(\beta - 1)! \left(\frac{a}{2}\right)^{1-\beta} .t^{\beta-\alpha} .p^{\frac{\beta}{2}-\alpha+\frac{1}{2}} J_{\beta-1}\left(ap^{1/2}\right) \quad (62)$$

Where,

$$p^n = \frac{t^n}{(-n)!} \quad (63)$$

Where, $F(\alpha, \beta, x)$ is the confluent Hyper geometric functions and is defined as

$$F(\alpha, \beta, x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\beta)!} x^k \quad (64)$$

By the transformations (62), the solution (60) and (61) becomes

$$W_a(\mu) = \sqrt{\mu}\, e^{\frac{\mu}{2}} \left[J_{-2m}(2\sqrt{\frac{\mu}{\pi}}) + R_3 J_{2m}(2\sqrt{\frac{\mu}{\pi}}) \right] \quad (65)$$

$$W_b(\mu) = \sqrt{\mu}\, e^{\frac{\mu}{2}} \left[J_{2m}(2\sqrt{\frac{\mu}{\pi}}) \right] \quad (66)$$

Then complete solution of the Equation (55) becomes

$$\xi_a(z) = \frac{e^{\frac{\mu}{2}}}{\sqrt{\mu}} \left[J_{-2m}(2\sqrt{\frac{\mu}{\pi}}) + R_3 J_{2m}(2\sqrt{\frac{\mu}{\pi}}) \right] \quad (67)$$

$$\xi_a(\mu\mu = \frac{e^{\frac{\mu}{2}}}{\sqrt{\mu}} \left[J_{2m}(2\sqrt{\frac{\mu}{\pi}}) \right] \quad (68)$$

The solution (49) using standard tables, Erdyle et al. (1954), Schaum Series (1950) Beck et al. (1992), and the inverse Laplace transform can be carried out for the solutions assuming $X_o = 0$ we get,

$$\hat{C}(p,z,s) = -\frac{\overline{W(s)} e^{\left(\frac{\mu+\mu_o}{2}\right)}}{3\sqrt{\mu\mu_o}} \begin{cases} \dfrac{\left[J_{-2m}(2\sqrt{\frac{\mu}{\pi}}) + R_3 J_{2m}(2\sqrt{\frac{\mu}{\pi}})\right] J_{2m}(2\sqrt{\frac{\mu_o}{\pi}})}{A_3}, & z<z_0 \\ \dfrac{\left[J_{-2m}(2\sqrt{\frac{\mu_o}{\pi}}) + R_3 J_{2m}(2\sqrt{\frac{\mu_o}{\pi}})\right] J_{2m}(2\sqrt{\frac{\mu}{\pi}})}{A_3}, & z>z0 \end{cases} \quad (69)$$

Where R_3 can be determined through on of the boundary conditions at z= 0, and is found to be

$$R_3 = -\left(\frac{\pi^2}{4p}\right)^m \left[\frac{2\sqrt{p} - 2m - 1}{2\sqrt{p} + 2m - 1}\right] \quad (70)$$

and A_3 is determined though Wronskian and is found to be

$$A_3 = -m \frac{e^{2\sqrt{p}(1-zo)}}{\sqrt{p}} \quad (71)$$

Then Equation (69) becomes by assuming that the concentration is far from the source, that is, distance $x >> 1$ or $p \to 0$. Then the Bessel's function is approximated

$$\hat{C}(p,z,s) = -\frac{\overline{W(s)}}{3\left(-m\frac{e^{2\sqrt{p}(1-zo)}}{\sqrt{p}}\right)} \begin{cases} \dfrac{e^{\mu/2}}{\sqrt{\mu}}\left[\left(\frac{\mu}{\pi}\right)^{-m} + R_3\left(\frac{\mu}{\pi}\right)^{m}\right] \dfrac{e^{\mu_o/2}}{\sqrt{\mu_o}}\left(\frac{\mu_o}{\pi}\right)^{m}, & z<z_0 \\ \dfrac{e^{\mu_o/2}}{\sqrt{\mu_o}}\left[\left(\frac{\mu_o}{\pi}\right)^{-m} + R_3\left(\frac{\mu_o}{\pi}\right)^{m}\right] \dfrac{e^{\mu/2}}{\sqrt{\mu}}\left(\frac{\mu}{\pi}\right)^{m}, & z>z_0 \end{cases} \quad (72)$$

$$\hat{C}_1(p,z,s) = \overline{W(s)} \frac{e^{\sqrt{p}(z_o - z)}}{6\sqrt{(1-z)(1-z_o)}} \left[\frac{e^{-mh_1}}{m} - \left(\frac{\sqrt{p} - m - 1/2}{\sqrt{p} + m - 1/2}\right)\frac{e^{-mh_2}}{m}\right], \quad z < z_0 \quad (73)$$

Where,

$$h_1 = \log\left(\frac{1-z}{1-z_o}\right) \quad (74)$$

$$h_2 = \log\left(\frac{1}{(1-z)(1-z_o)}\right) \quad (75)$$

$$m=\sqrt{\frac{1}{4}+p+s+\alpha} \tag{76}$$

Taking inverse Laplace transform along s, we get

$$\bar{c}_1(p,z,s)=\frac{e^{\sqrt{p}(z_o-z)}}{6\sqrt{(1-z)(1-z_o)}}L^{-1}\left\{\overline{W(s)}\left[\frac{e^{-mh_1}}{m}-\left(\frac{\sqrt{p}-m-1/2}{\sqrt{p}+m-1/2}\right)\frac{e^{-mh_2}}{m}\right]\right\} \tag{77}$$

$$L^{-1}\left\{\overline{W(s)}\right\}=W(t)=U(t-t_o) \tag{78}$$

$$L^{-1}\left\{\frac{e^{-mh_1}}{m}-\left(\frac{\sqrt{p}-m-1/2}{\sqrt{p}+m-1/2}\right)\frac{e^{-mh_2}}{m}\right\}=e^{-(\frac{1}{4}+\alpha+p)t}L^{-1}\left\{\frac{e^{-h_1\sqrt{s}}}{\sqrt{s}}-\left(\frac{\sqrt{p}-\sqrt{s}-1/2}{\sqrt{p}+\sqrt{s}-1/2}\right)\frac{e^{-h_2\sqrt{s}}}{\sqrt{s}}\right\}$$

$$=e^{-(\frac{1}{4}+\alpha+p)t}\left\{\frac{e^{-\frac{h_1^2}{4t}}+e^{-\frac{h_2^2}{4t}}}{\sqrt{\pi t}}-2\left(-\frac{1}{2}+\sqrt{p}\right)Erfc\left(\frac{h_1}{2\sqrt{t}}\right)\right\} \tag{79}$$

$$L^{-1}\left\{\overline{W(s)}\left[\frac{e^{-mh_1}}{m}-\left(\frac{\sqrt{p}-m-1/2}{\sqrt{p}+m-1/2}\right)\frac{e^{-mh_2}}{m}\right]\right\}$$

$$=\frac{e^{\frac{-z_0^2}{4x}}}{6\sqrt{\pi x^3}(1-z_0)}\left\{\left[K_1F_1(t)-K_2F_2(t)-\frac{F_3(t)}{2(\alpha+1/4)}\right]G_1(x,z_o)-z_o\left(K_3F_4(t)+K_4F_5(t)\right)\right\} \tag{80}$$

Where,

$$G_1(x,z_o)=zo\left(1+\frac{1}{x}\right)+2 \tag{81}$$

$$K_1(t)=\frac{e^{h_1\sqrt{\left(\frac{1}{4}+\alpha\right)}}}{4\alpha+1} \tag{82}$$

$$K_2(t)=\frac{e^{-h_1\sqrt{\left(\frac{1}{4}+\alpha\right)}}}{4\alpha+1} \tag{83}$$

$$K_3(t)=\frac{e^{h_1\sqrt{\left(\frac{1}{4}+\alpha\right)}}}{\sqrt{\alpha+\frac{1}{4}}} \tag{84}$$

$$K_4(t)=\frac{e^{-h_1\sqrt{\left(\frac{1}{4}+\alpha\right)}}}{\sqrt{\alpha+\frac{1}{4}}} \tag{85}$$

$$F_1(t)=Erfc\left(\frac{h_1}{2\sqrt{t}}+\sqrt{\left(\frac{1}{4}+\alpha\right)t}\right)-Erfc\left(\frac{h_1}{2\sqrt{t-t_o}}+\sqrt{\left(\frac{1}{4}+\alpha\right)(t-t_o)}\right) \tag{86}$$

$$F_2(t)=Erfc\left(\frac{h_1}{2\sqrt{t}}-\sqrt{\left(\frac{1}{4}+\alpha\right)t}\right)-Erfc\left(\frac{h_1}{2\sqrt{t-t_o}}-\sqrt{\left(\frac{1}{4}+\alpha\right)(t-t_o)}\right) \tag{87}$$

$$F_3(t)=e^{-(1/4+\alpha)t}Erfc\left(\frac{h_1}{2\sqrt{t}}\right)-e^{-(1/4+\alpha)(t-t_o)}Erfc\left(\frac{h_1}{2\sqrt{t-t_o}}\right) \tag{88}$$

$$F_4(t)=Erf\left(\frac{h_1}{2\sqrt{t}}+\sqrt{\left(\frac{1}{4}+\alpha\right)t}\right)-Erf\left(\frac{h_1}{2\sqrt{t-t_o}}+\sqrt{\left(\frac{1}{4}+\alpha\right)(t-t_o)}\right) \tag{89}$$

$$F_4(t)=Erf\left(\frac{h_1}{2\sqrt{t}}\sqrt{\left(\frac{1}{4}+\alpha\right)t}-\frac{h_1}{2\sqrt{t}}\right)-Erf\left(\sqrt{\left(\frac{1}{4}+\alpha\right)(t-t_o)}-\frac{h_1}{2\sqrt{t-t_o}}\right) \tag{90}$$

RESULTS AND DISCUSSION

This paper analyzed unsteady state dispersion of pollutants from a line source into stable atmosphere boundary layer. The quadratic diffusion coefficient and step-function type line source with variable wind profiles like constant, constant shear and parabolic wind are incorporated. We adopted multiple Laplace transform and Green's Function technique for the effective solution. In most of the previous models, numerical methods used either fully or partly. In model where the Laplace transform is generally inverted numerically (Robson, 1987). Whereas in this model, the multiple Laplace transform inverted analytically this gives an exact solution. The solutions of this model verified with that of (Robson, 1987) by assuming no delayed removal (chemical reaction rate) and time independent case. It is interesting to note that the effect of chemical reaction rte has virtually no impact t the short travel time but has significant effect t large travel time. It is interesting to note that when there is no chemical reaction of pollutant; the results of the present model coincide with the results of the model presented by Sulochana and Moka Shekhu (2009).

The results obtained in this model are illustrated graphically in the Figures 1 to 8. Concentration profiles of

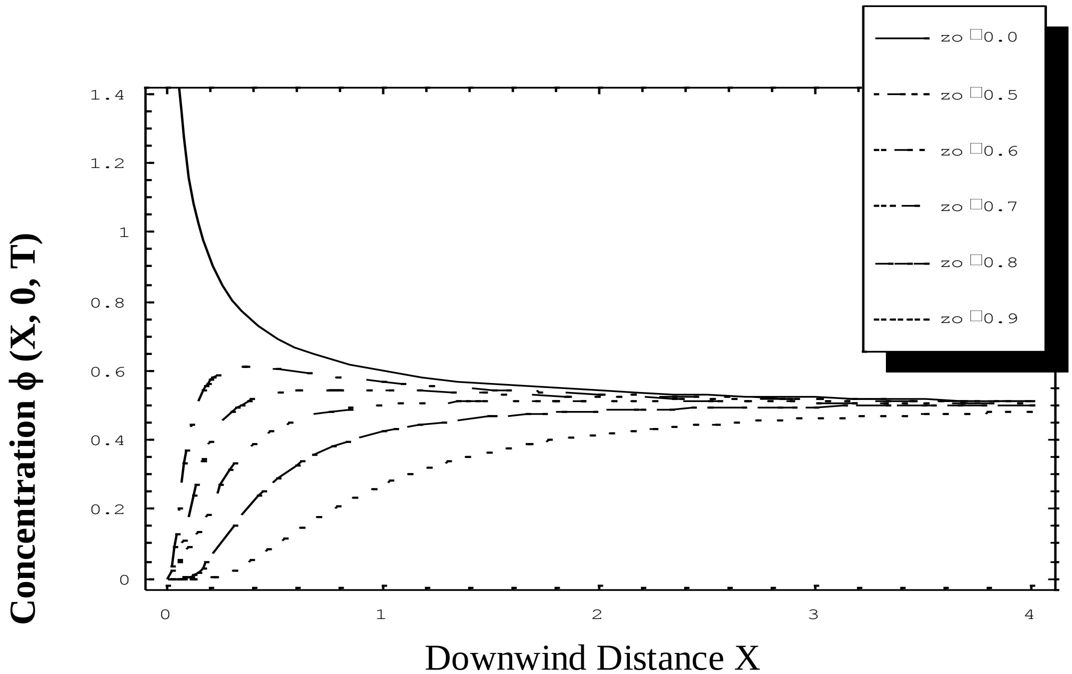

Figure 1. Ground level concentration for various source heights $z_o = 0, z_o = 0.5, z_o = 0.6, z_o = 0.7, z_o = 0.8, z_o = 0.9$ corresponding to the quadratic diffusion and Wind coefficient of constant shear. Comparing the results with Robson (1987) as limiting case where chemical reaction rate alpha=0.

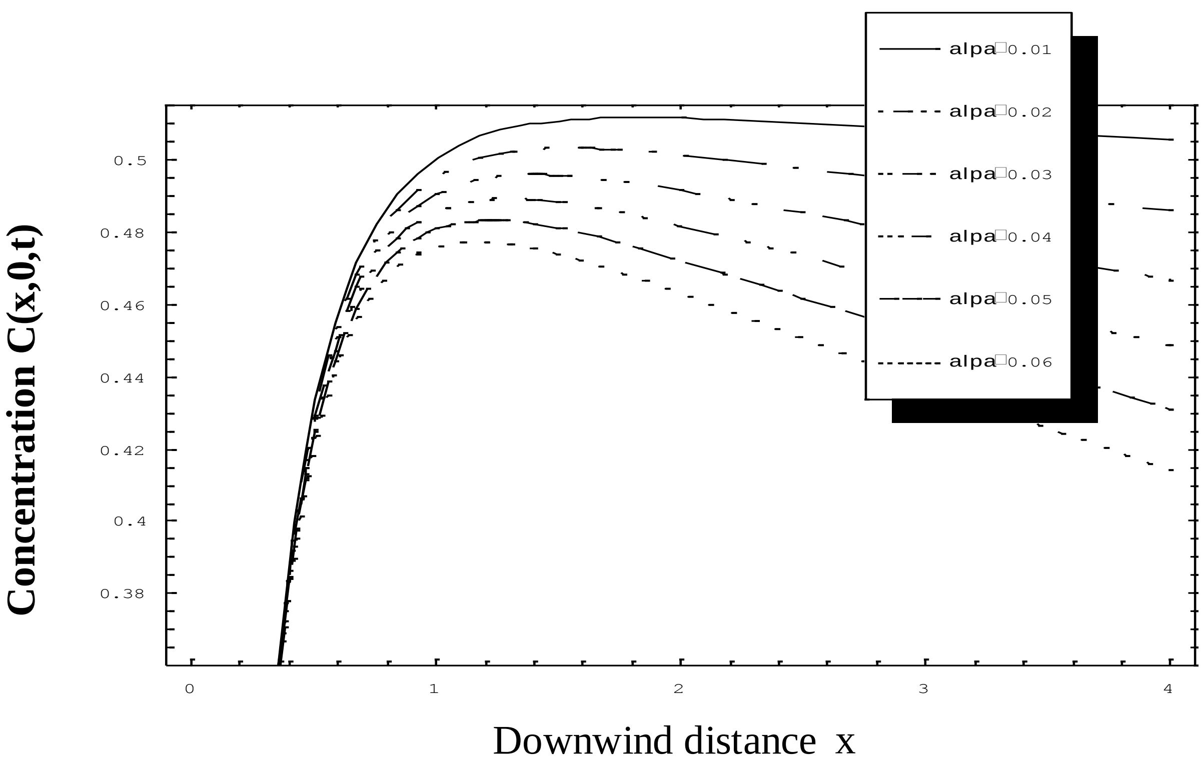

Figure 2. Ground level concentration for variable chemical reaction rate corresponding to the quadratic diffusion and wind coefficient of constant shear, $z_o = 0.0$.

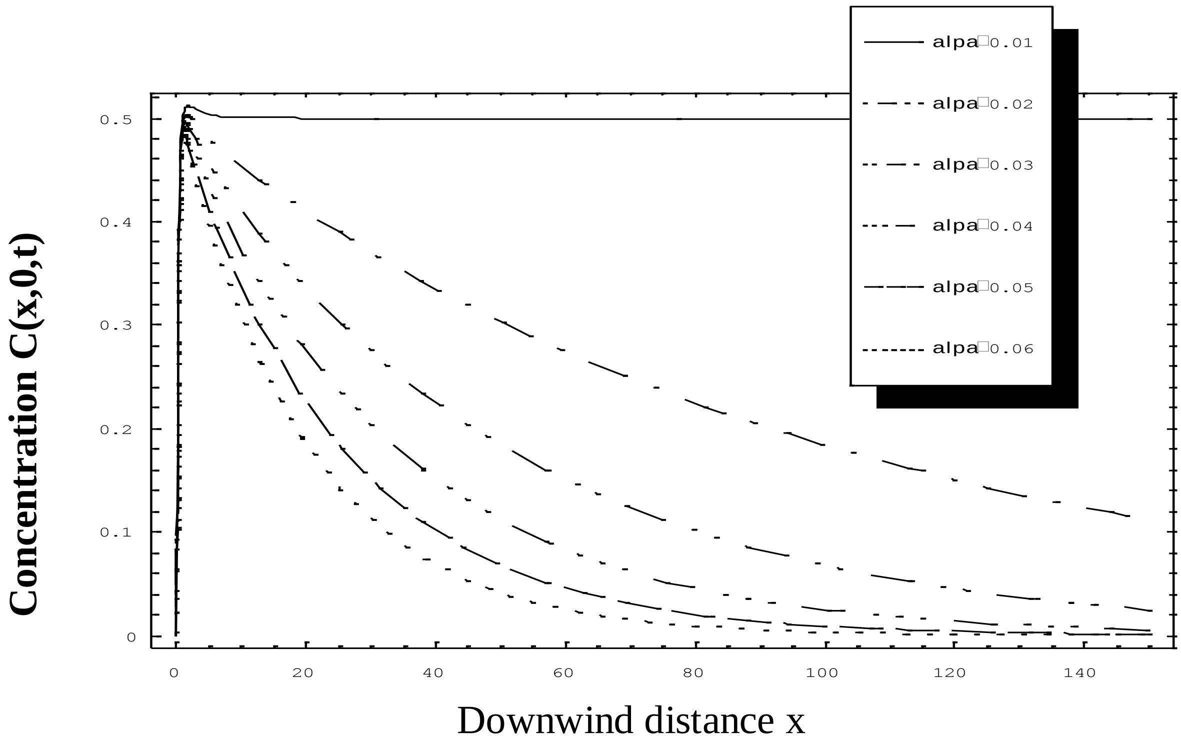

Figure 3. Ground level concentration for variable chemical reaction rate corresponding to the quadratic diffusion and wind coefficient of constant shear, $z_o = 0.7$.

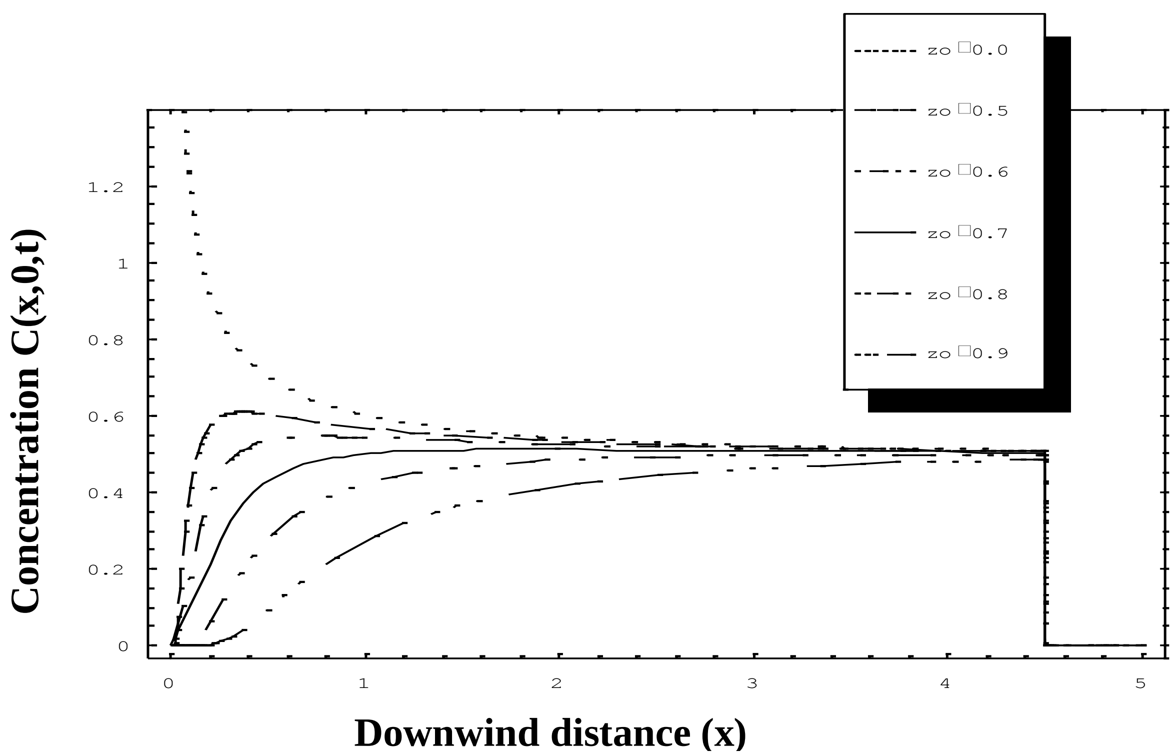

Figure 4. Ground level concentration vs. source heights corresponding to the quadratic diffusion and constant shear wind coefficient, chemical reaction rate $\alpha = 0.0$ at time $t = 5.0, \quad t_o = 0.5$.

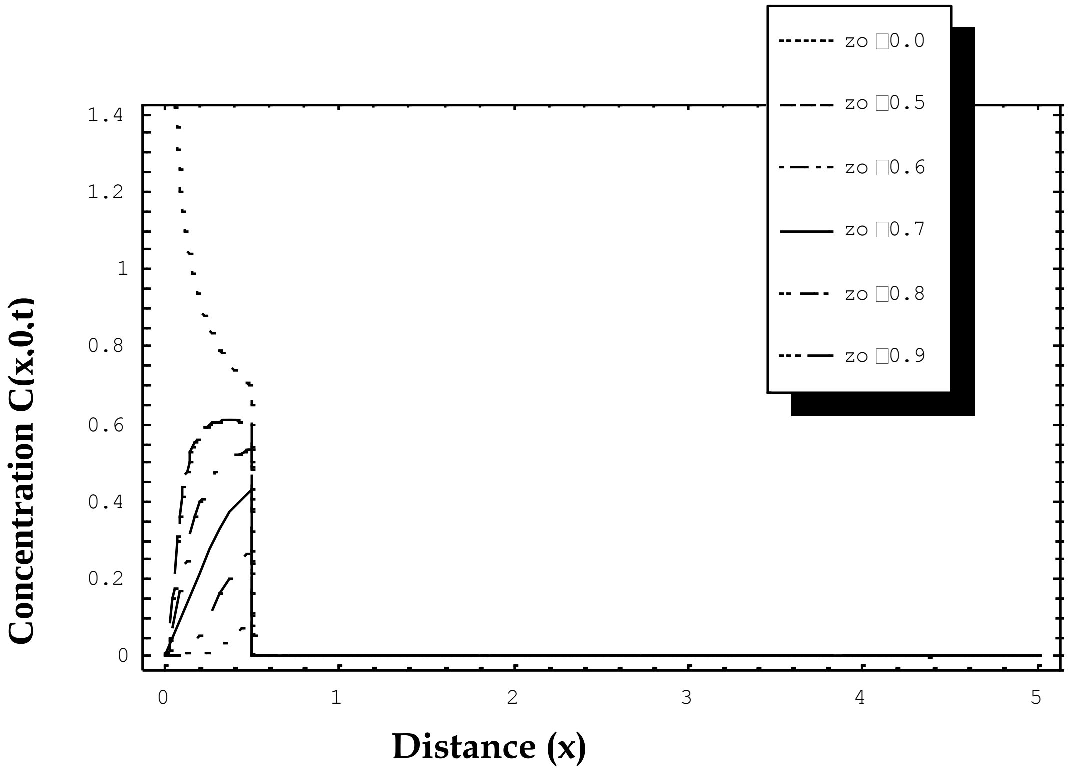

Figure 5. Ground level concentration versus source heights corresponding to the quadratic diffusion and constant shear wind coefficient, Chemical reaction rate $\alpha = 0.0$ at time $t = 1, \quad t_o = 0.5$

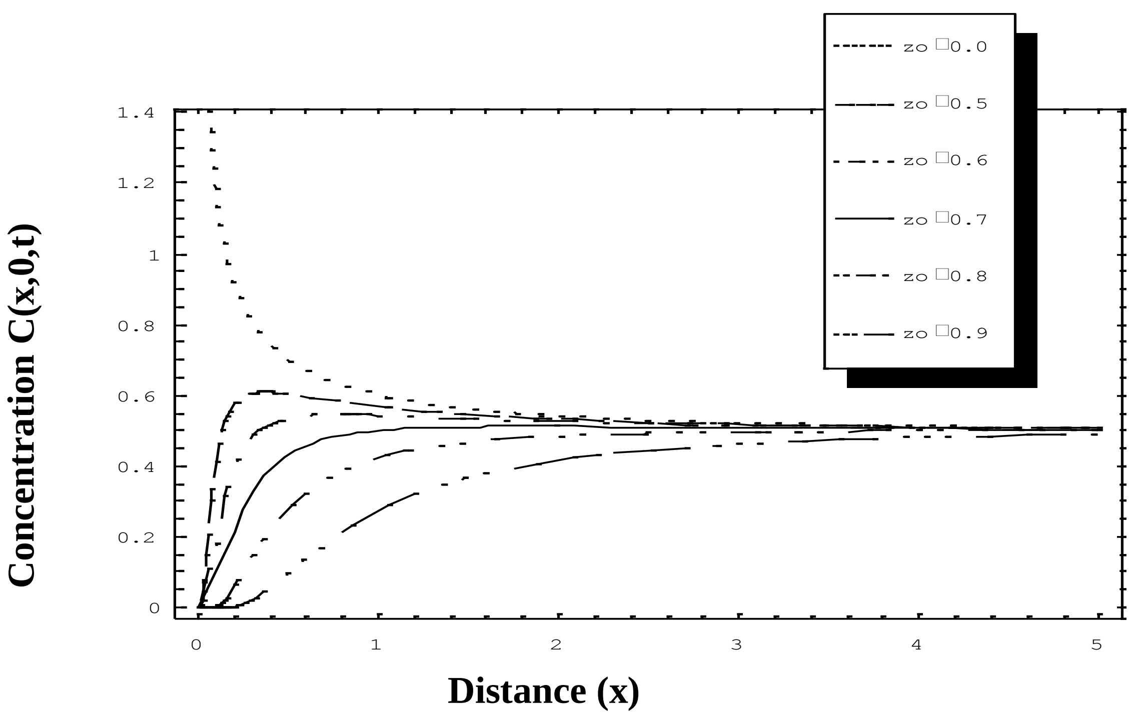

Figure 6. Ground level concentration versus source heights corresponding to the quadratic diffusion and constant shear wind coefficient, Chemical reaction rate $\alpha = 0.0$ at time $t = 15, \quad t_o = 0.5$

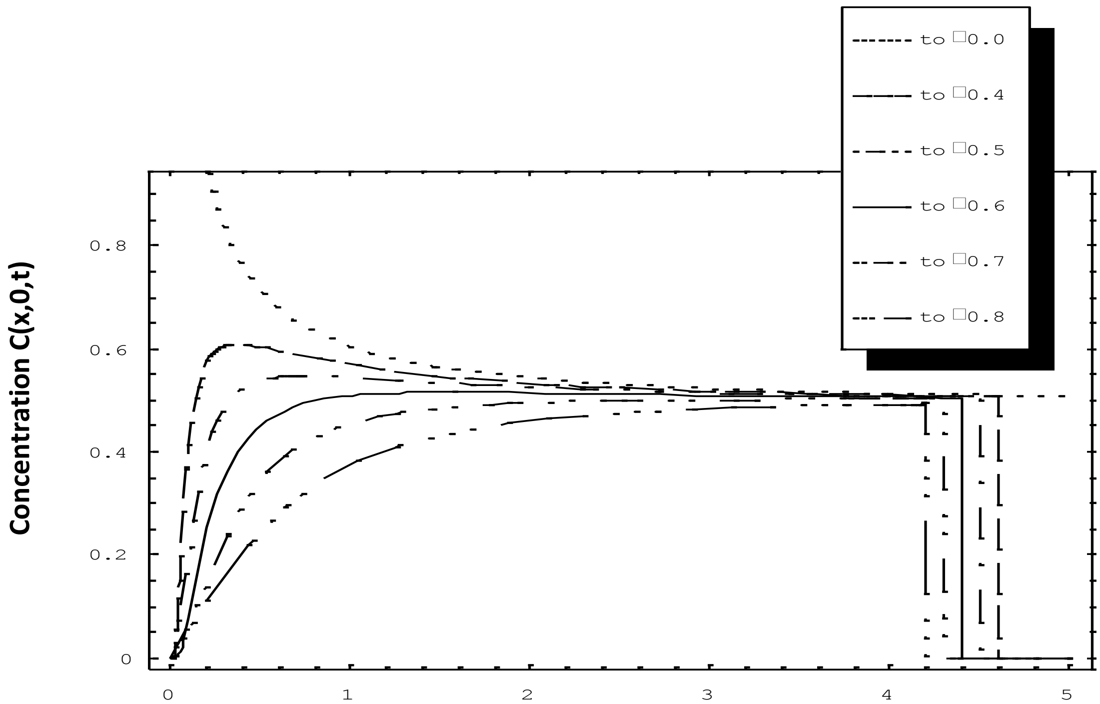

Figure 7. Ground level concentration versus downwind distance corresponding to the quadratic diffusion and constant shear wind coefficient, settling velocity for various time t_o Chemical reaction rate $\alpha = 0.0$ at time $t = 5, \quad z_o = 0.5$.

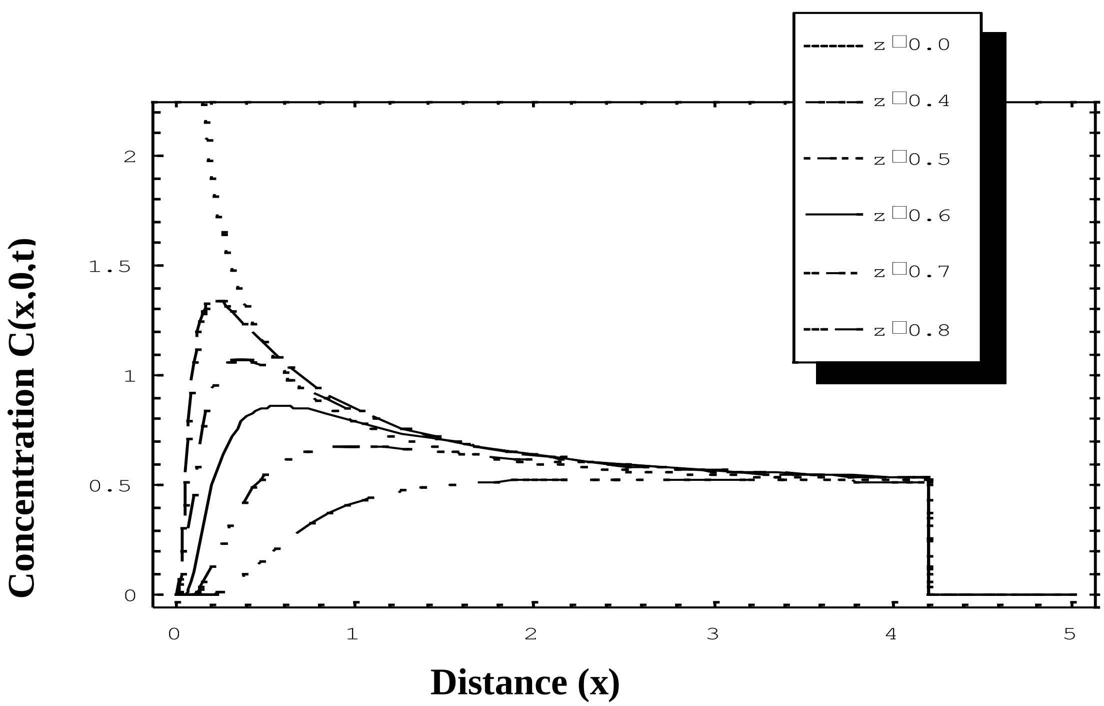

Figure 8. Ground level concentration versus distance for various heights on the surface to the quadratic diffusion and constant shear wind coefficient, chemical reaction rate $\alpha = 0.0$ at time $t = 5, \quad t_o = 0.5$

chemical reactions at large travel time are shown in Figures 2 to 8.

Nomenclatures

C (x, y, z) = Pollutant concentration (ppm)
H= height of the inversion layer (m)
z_o = height of the source (m)
(x_o, z_o) = the location of the source in xz -plan
K_z = Exchange coefficient along vertical direction
Q=Source strength at (x_o, z_o)
δ(.)= Dirac Delta function
K''=First order delayed removal
t = time in seconds
W (t) = time dependent source of step-function type and is of the form:
α = is the non-dimensional chemical reaction.

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