Review

Optimal control theory and fishery model

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Adequate numbers of relations have been provided to find the three unknowns following three equations: the state equation, the adjoint equation and maximum principle equation. If rigor is sacrificed, then a partial solution is quickly obtained by using the concept of calculus of variation. Our appealing and intuitive harvesting policy would be that, refrain from harvesting along the singular path, because zero harvest is not optimal.

Key words: Harvesting, bang-bang-control, singular control, maximum principle hamiltonian, Euler equation and switching function.

INTRODUCTION

A large cross-section of contemporary problems in applied mathematics, related to Biology is concerned with the analysis and synthesis of dynamic processes. The structural stability of a dynamic system depends on the parameters or structural constants appearing in the system of differential equations describing the system. During the last three decades, the management of natural resources in general and that of renewable resources, in particular, has invited the attention of a large segment of researcher (Goundry, 1960; Crutchfield, 1967; Wat, 1968; Garrod, 1973; Gulland, 1974). Coyle studied the dynamics of management system (Coyle, 1977) and of capital expenditure (Coyle, 1979). If $h(t)$ represents the rate of removal or harvesting then the population growth with harvesting is described by the differential equation

$$\frac{dn}{dt} = f(n) - h(t) \qquad (1)$$

where $n(t)$ denote the size of a fish population at time t. Whenever the harvest rate $h(t)$, exceeds the natural growth rate $f(n)$, Equation (1) implies that the population level will decline as $\frac{dn}{dt}$ becomes negative. However, if $h(t) < f(n)$, then the population growth continue. If $h(t) = f(n)$, the population remains at a constant level. Thus, in this situation, the natural growth rate $f(n)$ becomes the 'sustainable yield' that can be harvested while maintaining the population at a fixed level. Symbolically, the sustainable yield Y will be given by:

$$Y = f(n) = En, \qquad (2)$$

where E is the effort per-unit catch. For density dependent growth models degree $f(n) \geq 2$, therefore, if h is constant and $h < \max f(n)$, then Equation (1) may possess two or more equilibriums. An explicit analysis of the model can be carried out only when $f(n)$ is given in explicit form. However, if $h(t) = h$, then Equation (1) implies that a maximum sustainable yield (MSY) is given

by

$$Y_{\max} = h_{MSY} = \max_{n} f(n) \qquad (3)$$

with the property that any larger harvest rate will result into the depletion, and hence eventual extinction of the population. In order to achieve the maximum revenue return from fish harvesting and also to determine an optimal policy for fish harvesting, Pontryagin's maximum principle have applied. In this direction, further, if we assume a constant price.

p, per-unit of harvested bio-mass, and a constant cost c, per-unit catching effort, then the total sustainable revenue TR and total fishing cost TC are given by

$$TR = pY(E) \qquad (4a)$$

and

$$TC = cE \qquad (4b)$$

The net revenue, which is the difference TR and TC is called the 'sustainable economic rent'. Thus

$$SR = TR - TC = p.Y(E) - cE . \qquad (4c)$$

Gordon (1954) fundamental result state that, in the open-access fishery, effort tends to reach an equilibrium, the so-called bionomic equilibrium, at the level $E = E_{\infty}$, at which the sustainable economic rent is completely dissipated, that is

$$TR = TC \qquad (5)$$

In Gordon's model of open-access-fishery, if $E > E_{\infty}$ then opportunity cost exceeds revenues, consequently fishermen leave the fishery. Conversely, if $E < E_{\infty}$, then revenues exceed opportunity costs and consequently efforts tend to increase, as now fishing is more profitable than other employment (Clark, 1990; Burghes and Graham, 1980). At this point a reasonable inquiry is: what is wrong with a situation in which fishermen earn their exact opportunity cost from fishing? A close scrutiny shows that, firstly, the fishery resource which is capable of producing positive economic rent, for an excessive level of effort is being utilized. Neither the fishermen, nor society at large, are enjoying the benefits that could accrue as when the fisheries were under management. This situation is called 'economic overfishing'. Secondly, the fishery may suffer from 'biological overfishing' in the sense that in this case sustained biomass yield is less than MSY.

MODELS

Shah and Sharma (2003) proposed a deterministic extension of Gordon-Schaefer (GS) model by setting

$$\frac{dn}{dt} = rn\left[1 - \left(\frac{n}{K}\right)^{\alpha-1}\right] - En , \qquad (6)$$

where $n(t)$ is the stock size, $r > 0$ is the intrinsic growth rate per unit, K is the carrying capacity of the system, E is the effort per unit catch, and α is a real positive number exceeding 1, that is $\alpha > 1$. The model encompasses the following three models, which have been extensively pursued in the management of fishery (Pella and Tomlinson, 1969; Holt, 1975).

(i) Gordon-Schaefer (GS) model with $\alpha = 2$,
(ii) Pella-Tomlinson model (PT) with $\alpha = 3$ and
(iii) Pella-Tomlinson model (PT) with $\alpha = 4$.

MAXIMUM PRINCIPLE AND OPTIMAL HARVESTING

Considering the concept of opportunity cost, the maximum sustainable yield may not be profitable economically. Now we shall re-examine the model from economic perspective. Usually the harvest rate is determined by the current stock size $n(t)$, and the rate of harvesting effort E. Therefore we can write

$$h(t) = Q(n, E). \qquad (7)$$

The function $Q(n, E)$, which relates the factor of production n and E to the rate of production $h(t)$ is referred to as the production function. In our problem, we shall consider $Q(n, E)$ in the form:

$$Q(n, E) = G(n).E \qquad (8)$$

The linearity in effort E, facilitates the application of the maximum principle to our model; therefore, the reasons for this choice are primarily mathematical. $G(n)$, in view of physical aspect, is any non-decreasing function of n. Next, suppose the price p per-unit bio-mass remains constant, and that the cost c of a unit of effort is also constant. The net economic revenue P produced by an input of effort E over unit time will be given by

$$P = p.h(t) - c.E \qquad (9)$$

Combining Equations (7) to (9), we obtain

$$P = [p.G(n).E - c.E]$$
$$= [p - C(n)]h(t)$$

(10)

Where

$$C(n) = \frac{c}{G(n)}.$$

Now suppose that the sole owner's objective is to maximize the total discounted net revenue (the present value) $J(h)$, derived from harvesting of the fish population over finite horizon $[0,T]$, and given that

$$J(h) = \int_0^T e^{-\delta t} . P \, dt = \int_0^T e^{-\delta t} . [p - C(n)]h(t) \, dt$$

(11)

where $\delta > 0$ is a constant denoting the continuous discounting rate. In Equation (11) $h(t)$ may be viewed as a control variable, in conjunction with the constraint

$$h(t) = rn\left[1 - \left(\frac{n}{K}\right)^{\alpha-1}\right] - \frac{dn}{dt}$$

(12)

obtained from Equation (6). Combining Equation (11) and (12), our problem reduces to:

$$\max imize \left\{ J(h) = \int_0^T e^{-\delta t} [p - C(n)] \left[rn\left(1 - \left(\frac{n}{K}\right)^{\alpha-1}\right) - \frac{dn}{dt} \right] dt \right\}.$$

(13)

It will be worth mentioning that if we sacrifice the rigor, then a partial solution can be quickly obtained by using the ideas of calculus of variation (Gelfand and Fomin, 1961; Elsgolts, 1970; Bolza, 1951; Weinstock, 1974). Functional $J(h)$ in Equation (13) is analogous to the functional

$$I(x) = \int_{x_0}^{x_T} g(t, x; \dot{x}) dt$$

related to a variation problem seeking a path $x*$ from point x_0 to x_1 in a plane along which $I(x*)$ becomes maximum/minimum, depending on the nature of the problem (Maunder, 2002; Huo et al., 2012). Obviously, a necessary condition is that the path $x(t)$ must satisfy the classical Euler equation.

$$\frac{\partial g}{\partial x} = \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right)$$

(14)

In our problem an analog of the integrand is

$$g[t, n(t), \dot{n}(t)] = e^{-\delta t}[p - C(n)][f(n) - \dot{n}]$$

(15)

Therefore, n(t) must be a solution to

$$\frac{\partial g}{\partial n} = \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{n}}\right)$$

or

$$e^{-\delta t}\left[-\frac{dC(n)}{dn}\{f(n) - \dot{n}\} + \{P - C(n)\}\frac{df}{dn}\right] = \frac{d}{dt}\left[-e^{-\delta t}\{P - C(n)\}\right]$$

(16)

On simplification Equation (16) reduces to

$$-\frac{dC(n)}{dn}f(n) + \{P - C(n)\}\frac{df}{dn} = \delta[P - C(n)]$$

or

$$\frac{df}{dn} - \left[\frac{f(n)}{P - C(n)}\right]\frac{dC}{dn} = \delta$$

(17)

Equation (17) is an implicit equation describing the growth curve of the population yielding maximum economic revenue. If $n*$ is the unique solution to Equation (17), then given an initial population $n(0) = n_0$, the optimal harvest policy may be stated as follows: Utilize the harvest rate $h*(t)$, that drives the population level $n = n(t)$ towards $n*$ as rapidly as possible. If h_{max} represents the maximum feasible harvest rate, then we have,

$$h*(t) = \begin{cases} h_{max} & for \quad n > n* \\ f(n*) & for \quad n = n* \\ 0 & for \quad n < n* \end{cases}$$

(18)

In what follows, we shall apply Pontryagin's maximum principle (Pontryagin et al., 1962) for optimal control theory.

Pontryagin's MAXIMUM PRINCIPLE

Consider the differential equation

$$\frac{dn(t)}{dt} = f_0[t, n, h(t)],$$

(19)

with initial condition

$$n(0) = n_0 \qquad (20)$$

where $f_0(t,n,h)$ is a continuously differentiable function of three variables t, n and h. The variable $n(t)$, which describes the state of system at time t, will be called the 'state variable', and the Equation (19) will be referred to as the 'state equation', and $h(t)$ as the 'control function'. Further the terminal time T will be called the 'time horizon', and may become infinite in a problem. Any piece-wise continuous real-valued functions $h(t)$ defined for $0 \le t \le T$ will be called an admissible control. For a given admissible control $h(t)$, the solution to Equation (19) will be called the 'response'. Finally, the condition

$$n(T) = n_T \qquad (21)$$

will be termed as the 'terminal control', and a feasible control is one for which the response satisfies both the initial as well as the terminal condition. Now suppose that our objective functional is

$$J(h) = \int_0^T g[t,n(t),h(t)]dt \qquad (22)$$

where $g(t,n,h)$ is a given continuously differentiable function and $n(t)$ denotes the response to the control function $h(t)$. The maximum principle is most conveniently described in terms of the so-called Hamiltonian H defined by setting

$$\begin{aligned} H &= H[t,n(t),h(t);\lambda(t)] \\ &= g[t,n(t),h(t)] + \lambda(t).f_0[t,n(t),h(t)] \end{aligned} \qquad (23)$$

where $\lambda(t)$ is an additional unknown function, and is called the 'ad joint' variable in the optimal control theory. We now state the Pontryagin's maximum principle (without proof): If $h(t)$ is an optimal control and $n(t)$ is the corresponding response, then there exists an adjoint variable $\lambda(t)$ such that the following equations are satisfied, for all t, $0 \le t \le T$:

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial n} = -\left[\frac{\partial g}{\partial n} + \lambda(t)\frac{\partial f_0}{\partial n}\right], \qquad (24)$$

$$H[t,n(t),h(t);\lambda(t)] = \max_{h(t)}[H(t),n(t),h(t);\lambda(t)]. \qquad (25)$$

The maximization is carried out over all admissible controls $h(t)$. It is pertinent to note that Equation (25), factually tells that

$$\frac{\partial H}{\partial h} = 0. \qquad (26)$$

Now the question is how to apply this principle to a concrete problem? Here we have three unknowns $n(t)$, $h(t)$ and $\lambda(t)$. For these three functions, we have three equations, namely: the state Equation (6) for $n(t)$, the adjoint Equation (24), and the maximum principle Equation (25) equivalently Equation (26). Furthermore, we have initial condition Equation (20), and the terminal condition Equation (21). Thus, in principle, adequate number of relations have been provided to find the unknown functions $n(t)$, $h(t)$ and $\lambda(t)$. In case of our problem it is not possible to provide the terminal condition $n(T)$. But the function $J(h)$, given by Equation (22) can be written as

$$J[h(t)] = \int_0^T \left\{ e^{-\delta t}[p-C(n)]\left[rn\left(1-\left(\frac{n}{K}\right)^{\alpha-1}\right)\right] - e^{-\delta t}[p-C(n)]\frac{dn}{dt} \right\} d(t) \qquad (27)$$

or

$$J(h) = \int_0^T \left[A(n,t) + B(n,t)\frac{dn}{dt} \right].dt \qquad (28)$$

Where

$$A(n,t) = e^{-\delta t}[p-C(n)]\left[rn\left(1-\left(\frac{n}{K}\right)^{\alpha-1}\right)\right]$$

$$B(n,t) = -e^{-\delta t}[p-C(n)] \qquad (29)$$

and $\dfrac{dn(t)}{dt}$ remains bounded for all times, that is,

$$A(n,t) \le \frac{dn}{dt} \le B(n,t). \qquad (30)$$

Therefore, if we introduce

$$\frac{dn}{dt} = h(t). \qquad (31)$$

Then the Hamiltonian of the problem becomes

$$H = (A + Bh) + \lambda(t)h$$
$$= A + (B + \lambda)h \quad . \tag{32}$$

According to the maximum principle Equation (25), the optimal control $h(t)$ must maximize H in Equation (32). If we define

$$\psi(t) = B(n,t) + \lambda(t), \tag{33}$$

then $h(t)$ must satisfy

$$h(t) = \begin{cases} B(n,t) & if \quad \psi(t) > 0, \\ A(n,t) & if \quad \psi(t) < 0. \end{cases} \tag{34}$$

A control like $h(t)$, which assumes these extreme values (condition Equation 30) is called a 'bang-bang' control, and for obvious reason $\psi(t)$ is called the 'switching function'. Whenever $\psi(t)$ vanishes, then

$$H = A(n,t), \tag{35}$$

that is, Hamiltonian is independent of the control $h(t)$, and consequently the maximum principle does not specify the value of optimal control. The most remarkable case, the so called singular case, arise when $\psi(t)$ vanishes identically over some time interval of positive duration; thus, if

$$\psi(t) = B(n,t) + \lambda(t) \cong 0 \tag{36}$$

then the corresponding singular control $h(t)$ is determined as follows. Equation (36) yields

$$\frac{d\psi}{dt} = \frac{\partial B}{\partial n} \cdot \frac{dn}{dt} + \frac{\partial B}{\partial t} + \frac{d\lambda}{dt}$$
$$= \frac{\partial B}{\partial n} h + \frac{\partial B}{\partial t} - \left(\frac{\partial H}{\partial n} \right)$$
$$= \frac{\partial B}{\partial n} h + \frac{\partial B}{\partial t} - \frac{\partial}{\partial n} [A + (B + \lambda)h] \tag{37}$$
$$= \frac{\partial B}{\partial n} h + \frac{\partial B}{\partial t} - \frac{\partial A}{\partial n} - \frac{\partial B}{\partial n} h$$
$$= \frac{\partial B}{\partial t} - \frac{\partial A}{\partial n} = 0$$

But Equation (37) can be directly derived from the Lagrange's equation of the variational problem Equation (28). Hence Equation (37) is the equation of the singular path

$$n = n^*(t). \tag{38}$$

Therefore, $\psi(t) \cong 0$ corresponds to the singular solution given by Equation (38). Thus the maximum principle implies that the optimal control $h = \dfrac{dn}{dt}$ for a linear problem must be a combination of 'bang-bang' and a 'singular control'.

APPLICATION OF THE MAXIMUM PRINCIPLE TO FISHERY PROBLEM

For our problem the Hamiltonian H becomes:

$$H = e^{-\delta t}[p - C(n)]h(t) + \lambda(t)\left[rn\left(1 - \left(\frac{n}{K} \right)^{\alpha-1} \right) - h(t) \right]. \tag{39}$$

Therefore, the switching function is given by

$$\psi(t) = e^{-\delta t}[p - C(n)] - \lambda(t). \tag{40}$$

Consequently, the singular path $\psi(t) = 0$, gives

$$\lambda(t) = e^{-\delta t}[p - C(n)]. \tag{41}$$

But this is precisely what is obtained on setting $\dfrac{\partial H}{dh} = 0$.

Thus,

$$\frac{\partial H}{dh} = e^{-\delta t}[p - C(n)] - \lambda(t).$$

or

$$\lambda(t) = e^{-\delta t}[p - C(n)]$$

which is precisely Eq.(41). When $n(T)$; is not specified, we invoke the 'free terminal-value condition', which reads

$$\lambda(T) = 0 \tag{42}$$

Now in our problem, since $p > C(n^*)$ the free-terminal-value condition Equation (42) implies that we must leave the singular path $n = n^*$ before $t = T$, while off the singular path we must use a 'bang-bang' control. Recalling that,

$$0 \le h(t) \le h_{max}, \tag{43}$$

and comparing Equation (43) with Equation (30), we get the optimal policy for harvesting via Equation (34), that is, as $h = 0$ is not optimal, and $h = h_{\max}$ for $t \leq T$, provides a positive contribution to present value. Therefore the policy should be:

(i) Singular path $h = 0$ for $t \leq t_0$,

(ii) Maximum harvest $h = h_{\max}$ for $t_0 < t < T$.

CONCLUDING REMARKS

In this paper, we have examined the generalization of Gordon-Schaefer fishery model from the economic perspective. In view of physical aspect, the harvest rate has been determined by the current stock size and the rate of harvesting effort. The linearity in effort facilitates the application of the maximum principle and for optimal control we have applied the Pontryagin's maximum principle. In order to solve the proposed non-linear fishery model we have used the more powerful optimization techniques provided by the calculus of variation. We derived the conditions under which the system will exhibit optimality. The optimal control implies that, we should leave the singular path before the time horizon while harvesting is maximum near the final time because it provides a positive contribution to the present values. However, time near to time horizon provides a positive contribution to the present value.

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