

Full Length Research Paper

Analysis of potential flow around two-dimensional body by finite element method

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The paper presents a numerical method for analyzing the potential flow around two dimensional body such as single circular cylinder, NACA0012 hydrofoil and double circular cylinders by finite element method. The numerical technique is based upon a general formulation for the Laplace's equation using Galerkin technique finite element approach. The solution of the systems of algebraic equations is approached by Gaussian elimination scheme. Laplace's equation is expressed in terms of both stream function and velocity potential formulation. A finite element program is developed in order to analyze the result. The contours of stream and velocity potential function are drawn. The contour of stream function exhibits the characteristics of potential flow and does not intersect each other. The calculated pressure co-efficient shows the pressure decreasing around the forward face from the initial total pressure at the stagnation point and reaching a minimum pressure at the top of the cylinder.

Key words: Stream function, velocity potential, number of nodes (NDE), number of elements (NEL).

INTRODUCTION

The flow past two dimensional body such as circular cylinders and hydrofoil has been the subject of numerous experimental and numerical studies because this type of flow exhibits the very fundamental mechanisms. The flow field over both the cylinder and hydrofoil is symmetric at low values of Reynolds number. As the Reynolds number increases, flow begins to separate behind the body causing vortex shedding which is an unsteady phenomenon. To achieve the goal of obtaining the detailed information of the flow field around two dimensional bodies, Finite Element Method (FEM) has been emerged as an attractive, powerful tool in many designing process.

The FEM was originated from the field of structural calculation in the beginning of the fifties and was introduced

by Turner et al. (1956). The FEM was introduced into the field of computational fluid dynamics (CFD) by Chung (1977). The first study concerning the steady flow past a circular cylinder was reported by Thom (1933) for Reynolds number of 10 and 20. The works of Kawaguti (1953) and Payne (1958) were restricted to low Reynolds numbers ($Re = 40$) and relatively low Reynolds numbers ($Re = 40\sim 100$) respectively. Oden (1969) has presented a theoretical finite element analogue for the Navier-Stokes equations, but without a practical numerical method. Dennis and Chung (1970) introduced finite element method into the field of computational fluid dynamics (CFD) by solving steady flow past a circular cylinder at Reynolds number ($Re \leq 100$).

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Tong (1971) presented results for steady flow using this method with pressure and velocities as dependent variables. Olson (1974) presented a numerical procedure to investigate steady incompressible flow problems using stream function formulation. Hafez (2004) simulates steady an inviscid flows over a cylinder using both potential and stream functions.

The objective of the present research is to analyze the potential flow around single circular cylinder, NACA 0012 hydrofoil and double circular cylinders by Galerkin technique of finite element method. Due to symmetry of circular cylinders and NACA 0012 hydrofoil, only the upper half portions have been considered as computational domain. Both stream function and velocity potential formulation have been used with definite boundary conditions. Contours of stream and velocity potential function, velocity above crest for both formulations and velocity and pressure distribution along the surface for various discretization are obtained which are compared with the analytical result available from the literature and shown graphically.

MATHEMATICAL FORMULATION

Potential flow around circular cylinder

Let us consider the potential flow of an ideal fluid around the circular cylinder placed with its axis perpendicular to the plane of the flow as shown in Figure 1. Now, the potential flow around circular cylinder can be represented by *Laplace* equation as:

$$\left. \begin{aligned} \nabla^2\Psi &= 0 \\ \nabla^2\Phi &= 0 \end{aligned} \right\} \quad (1)$$

The velocity components *u* and *v* of the flow field in relation to stream function Ψ or the velocity potential Φ are given by:

$$u = \frac{\partial\Psi}{\partial y}, v = -\frac{\partial\Psi}{\partial x} \quad \text{Or,}$$

$$u = \frac{\partial\Phi}{\partial x}, v = \frac{\partial\Phi}{\partial y} \quad (2)$$

Boundary conditions for stream function

The half of the fluid domain is taken in the computations as shown in Figure 2 and the boundary conditions that need to be satisfied in order to get the solution of *Laplace* equation: $\nabla^2\Psi = 0$ in Ω are given as follows:

- (a) $\Psi = 0$ on the boundary a-e-f-g
- (b) $\Psi = yU$ on the boundary a-b
- (c) $\Psi = yU$ on the boundary b-h
- (d) $\Psi = yU$ on the boundary g-h

Boundary conditions for velocity potential

In case of velocity formulation, the boundary conditions that need to be satisfied in order to get the solution of *Laplace* equation: $\nabla^2\Phi = 0$ in Ω as shown in Figure 3:

- (a) On the boundary a-b, $\frac{\partial\Phi}{\partial x} = U$
- (b) On the boundary a-e-f-g, $\frac{\partial\Phi}{\partial y} = 0$
- (c) On the boundary b-h, $\frac{\partial\Phi}{\partial y} = 0$
- (d) On the boundary g-h, $\frac{\partial\Phi}{\partial x} = U$

Potential flow around a hydrofoil

Let us consider the flow of an ideal fluid around a hydrofoil placed with its axis perpendicular to the plane of the flow as shown Figure 4.

Boundary conditions for stream function

Now we need to solve the *Laplace* equation: $\nabla^2\Psi = 0$ in Ω as shown in Figure 5 with the following boundary conditions:

- (a) $\Psi = 0$ on the boundary a-b-c-d
- (b) $\Psi = yU$ on the boundary a-f and e-d
- (c) $\Psi = yU$ on the boundary f-e

Boundary conditions for velocity potential

In case of velocity potential formulation, we need to solve the *Laplace* equation: $\nabla^2\Phi = 0$ in Ω as shown in Figure 6 with the following boundary conditions:

- (a) On the boundary a-f and d-e, $\frac{\partial\Phi}{\partial n} = 1$
- (b) On the boundary a-b-c-d and f-e, $\frac{\partial\Phi}{\partial n} = 0$

Potential flow around two circular cylinders

Let us consider potential flow around double circular cylinders as shown in Figure 7. The stream function for the flow can be expressed as

$$\Psi(x, y) = \Psi_1(x, y) + a\Psi_2(x, y) + b\Psi_3(x, y) \quad (3)$$

Where *a* and *b* are the two constants. Now we need to solve Laplace's equation

$$\nabla^2\Psi_1 = 0; \nabla^2\Psi_2 = 0; \nabla^2\Psi_3 = 0$$

with the following boundary conditions:

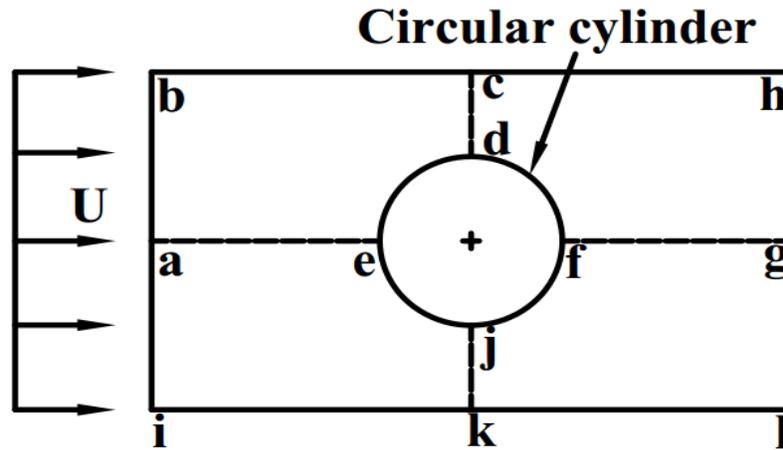


Figure 1. Flow around single circular cylinder.

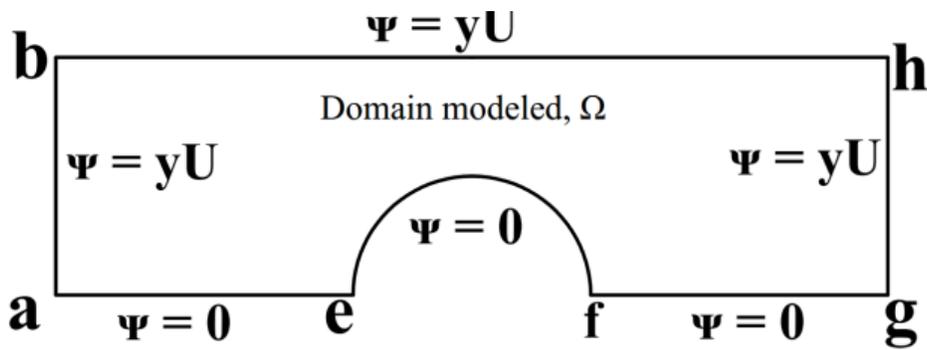


Figure 2. Boundary conditions for the stream function formulation.

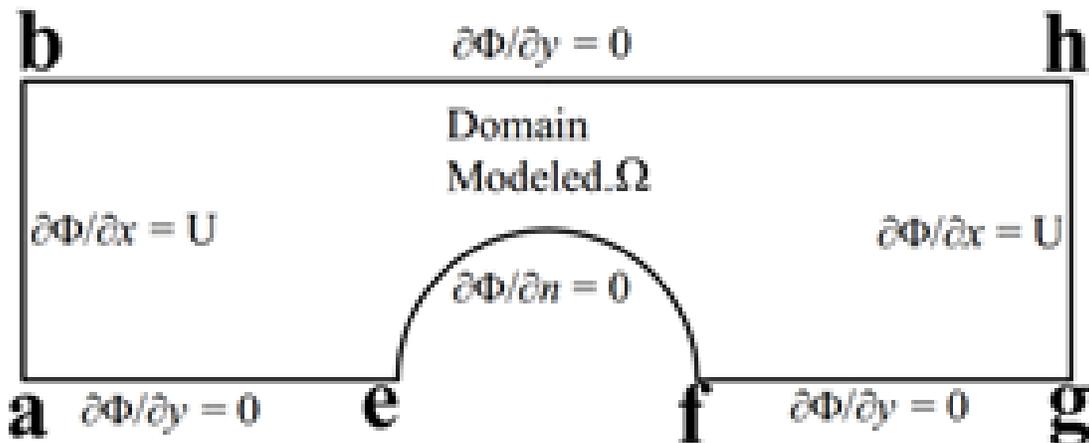


Figure 3. Boundary conditions for the velocity potential function formulation.

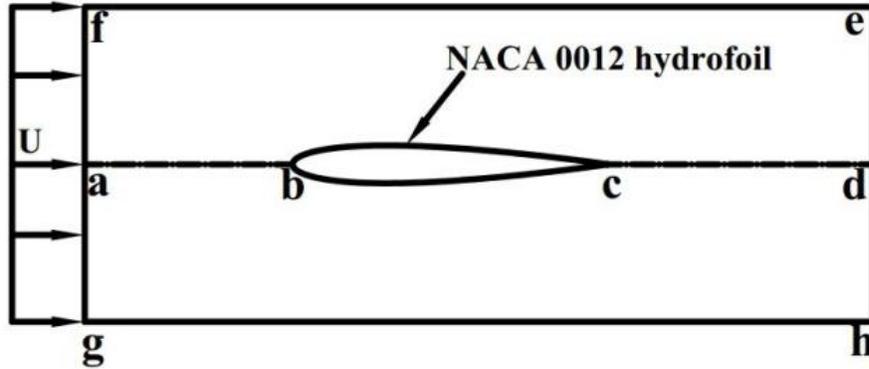


Figure 4. Flow around a NACA 0012 hydrofoil.

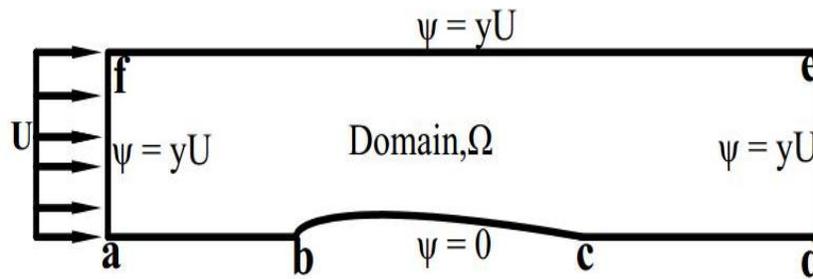


Figure 5. Boundary conditions for the stream function formulation for hydrofoil.

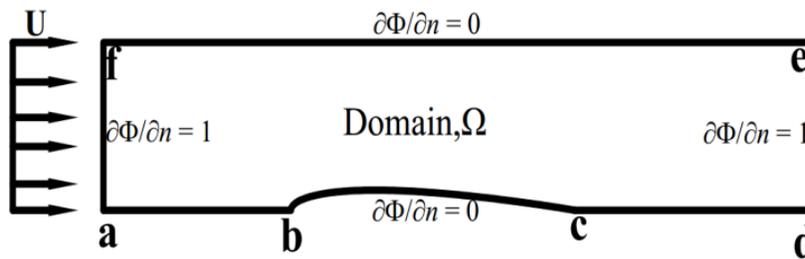


Figure 6. Boundary conditions for the velocity potential function formulation.

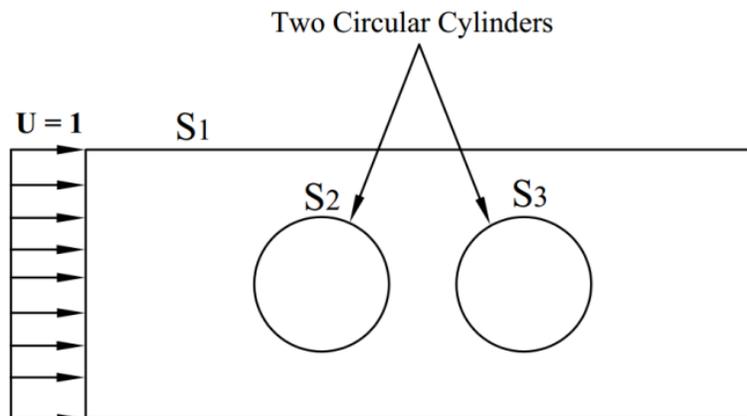


Figure 7. Flow around double circular cylinders with boundary conditions.

- (a) $\Psi_1 = U y$ on S_1
 (b) $\Psi_1 = 0$ on S_2 and S_3
 (c) $\Psi_2 = 0$ on S_1 and S_3
 (d) $\Psi_2 = 1$ on S_2
 (e) $\Psi_3 = 0$ on S_1 and S_2
 (d) $\Psi_3 = 1$ on S_3

NUMERICAL SOLUTION OF POTENTIAL FLOW

Numerical solution by stream function method

The stream function over the domain of interest is discretized into finite elements having M nodes:

$$\Psi(x, y) = \sum_{i=1}^M N_i(x, y) \Psi_i = [N] \{\Psi\} \quad (4)$$

Using the Galerkin method, the element residual equations are:

$$\int_{A^e} N_i(x, y) \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) dx dy = 0, i = 1, M \quad (5)$$

$$\text{or, } \int_{A^e} [N]^T \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) dx dy = 0 \quad (6)$$

Application of the Green-Gauss theorem gives

$$\begin{aligned} & \int_{S^e} [N]^T \frac{\partial \Psi}{\partial x} n_x dS - \int_{A^e} \frac{\partial [N]^T}{\partial x} \frac{\partial \Psi}{\partial x} dx dy + \int_{S^e} [N]^T \frac{\partial \Psi}{\partial y} n_y dS \\ & - \int_{A^e} \frac{\partial [N]^T}{\partial y} \frac{\partial \Psi}{\partial y} dx dy \end{aligned} \quad (7)$$

Where S represents the element boundary and (n_x, n_y) are the components of the outward unit vector normal to the boundary. Using Equation (4) in Equation (7) and substituting the velocity components into the boundary integrals, results in:

$$\begin{aligned} & \int_{A^e} \frac{\partial [N]^T}{\partial x} \frac{\partial [N]}{\partial x} + \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} dx dy \{\Psi\} \\ & = \int_{S^e} [N]^T (u n_y - v n_x) ds \end{aligned} \quad (8)$$

and this equation is of the form

$$[k^e] \{\Psi\} = \{f^{(e)}\} \quad (9)$$

The $M \times M$ element stiffness matrix is

$$[k^{(e)}] = \int_{A^e} \left(\frac{\partial [N]^T}{\partial x} \frac{\partial [N]}{\partial x} + \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} \right) dx dy \quad (10)$$

the nodal forces are represented by the $M \times 1$ column matrix

$$\{f^{(e)}\} = \int_{S^e} [N]^T (u n_y - v n_x) ds \quad (11)$$

Numerical solution by velocity potential method

The finite element formulation of potential flow of an ideal fluid in terms of velocity potential is quite similar to that of the stream function approach, since the governing equation is *Laplace's* equation in both cases. By direct analogy with Equations (4) to (11) it is obtained as follows:

$$\Phi(x, y) = \sum_{i=1}^M N_i(x, y) \Phi_i = [N] \{\Phi\} \quad (12)$$

Using the Galerkin method, the element residual equations are:

$$\int_{A^e} N_i(x, y) \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) dx dy = 0, i = 1, M \quad (13)$$

$$\text{or, } \int_{A^e} [N]^T \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) dx dy = 0 \quad (14)$$

Application of the Green-Gauss theorem gives

$$\begin{aligned} & \int_{S^e} [N]^T \frac{\partial \Phi}{\partial x} n_x dS - \int_{A^e} \frac{\partial [N]^T}{\partial x} \frac{\partial \Phi}{\partial x} dx dy + \int_{S^e} [N]^T \frac{\partial \Phi}{\partial y} n_y dS \\ & - \int_{A^e} \frac{\partial [N]^T}{\partial y} \frac{\partial \Phi}{\partial y} dx dy \end{aligned} \quad (15)$$

Utilizing Equation (12) in the area integral of Equation (15) and substituting the velocity components into the boundary integrals, results in:

$$\begin{aligned} & \int_{A^e} \frac{\partial [N]^T}{\partial x} \frac{\partial [N]}{\partial x} + \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} dx dy \{\Phi\} \\ & = - \int_{S^e} [N]^T (u n_x + v n_y) ds \end{aligned} \quad (16)$$

and this equation is of the form

$$[k^e] \{\Phi\} = \{f^{(e)}\} \quad (17)$$

RESULTS AND DISCUSSION

Based on the previous mathematical formulation as outlined as numerical solution by stream function method and numerical solution by velocity potential method a finite element program has been developed in FORTRAN 90 for calculating the potential flow around two dimensional bodies. For all finite element mesh configurations, nodes along the vertical line above the crest of the cylinder are numbered consecutively from top to bottom in order to be compatible with velocity calculations used in the program. The elements are taken in the form of triangle or quadrilateral for the convenience of discretization, thus the edge of the body may not be appeared as a circle or hydrofoil.

Single circular cylinder

Let us consider the flow around the circular cylinder of unit radius confined between two parallel plates having length of 7 m and height 4 m. A fluid of uniform velocity 1.0 m/s is assumed to be flowing from the left to the right of cylinder as shown in Figure 8. The choice of computational domain in the direction of flow is arbitrary and the free stream velocity is considered to prevail at distances sufficiently far from the cylinder. The upper half of the computational domain surrounded by the path (a-b-c-d-e-f) is taken into account for numerical calculation due to symmetry of flow and is discretized by (24×5) triangular elements as shown in Figure 9 for stream function formulation. The contour of stream function has been obtained from stream function formulation and exhibits the characteristics of potential flow as shown in Figure 10. The stream lines have not intersected each other and mean the flow past the cylinder smoothly without any separation at the trailing edge. The upper half of the computational domain for velocity potential formulation is also discretized by (20×7) quadrilateral elements as shown in Figure 11. The contours of velocity potential have also been obtained from velocity potential formulation and exhibit the characteristic of potential flow i.e. no vortices exist at the trailing edge as shown in Figure 12.

The velocities along the vertical line above the crest of the cylinder are calculated and then compared with the analytical result in Figure 13. The average deviation for the velocity profiles between the two cases is less than one percent. Figure 14 is plotted by calculating the velocities above the crest at two points ($x = 3.50, y = 1.00$) and ($x = 3.50, y = 2.00$) against various number of nodes for stream function formulation which shows that computed velocities converges to the analytical solution as number of nodes increases. Figure 15 is obtained by plotting the velocity square along the surface of cylinder against the angular coordinates of nodal points. There are two types of curves of which first type shows a sinusoidal curve for the whole cylinder obtained

theoretically and second type consists of four curves for four different finite element mesh configurations.

In Figure 16 the calculated pressure coefficient (C_p) is compared with the theoretical pressure distribution over the surface of the cylinder and the agreement is found to be quite satisfactory. The calculated results show the pressure decreasing around the forward face from the initial total pressure at the stagnation point and reaching a minimum pressure at the top of the cylinder.

NACA 0012 hydrofoil

Let us consider the flow around NACA 0012 hydrofoil confined between two parallel plates having length of 10 m and height 4m as shown in Figure 17. A fluid of uniform velocity 1 m/s is flowing from the left to right of the foil. The half of computational domain for stream function formulation is discretized by (16×3) elements as shown in Figure 18 and the contours of the stream lines are given in Figure 19.

Similarly, the half of computational domain for velocity potential formulation is discretized by (16×3) elements as shown in Figure 20 and the contours of the stream lines are given in Figure 21. Figure 22 depicts a comparison of pressure distribution pressure over the surface of the foil with the results obtained from constant strength source method¹¹ and shows very close agreement both at the leading and trailing edge of the foil.

Double circular cylinders

The flow around two circular cylinders of unit radius is confined between two parallel plates having length of 10 m and height 4m. The distance between the cylinders is unit length and the height above the cylinder is also unit length. A fluid of uniform velocity 1 m/s is flowing from left to right of cylinders as shown in Figure 23. The half of the computational domain is discretized by (16×3) elements for stream function formulation and (20×7) elements for velocity potential as shown in Figures 24 and 25, respectively. The contours obtained from these two formulations are drawn in Figures 26 and 27 respectively.

Conclusions

The paper presents a numerical method of calculating the potential flow around two dimensional bodies by finite element method. The following conclusions can be drawn from the present numerical analysis:

- (i) The present method can be an efficient tool for evaluating the potential flow characteristics of two dimensional body.
- (ii) The contour of stream function exhibits the characteristics of potential flow and does not intersect each other.

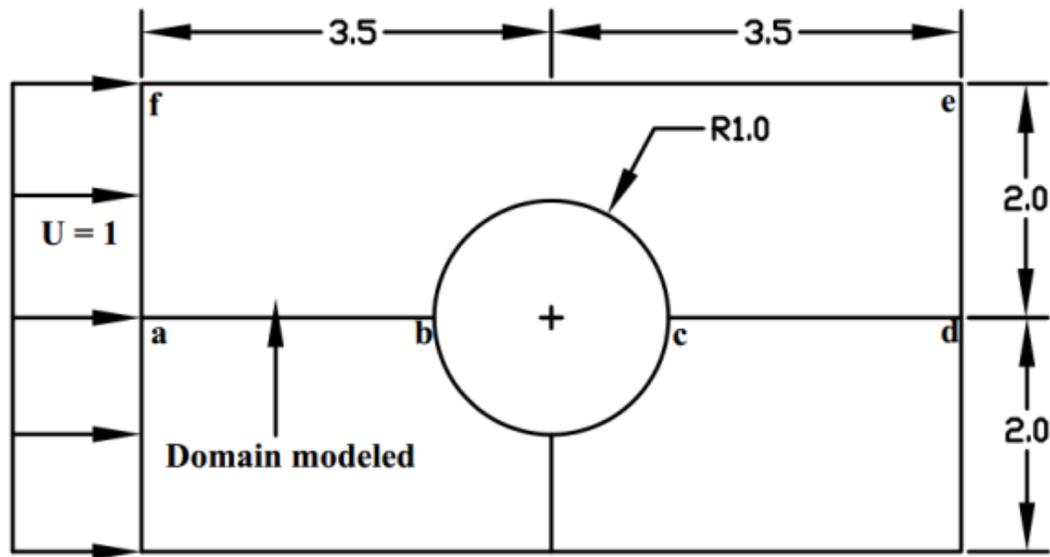


Figure 8. Computational domain for stream function and velocity potential formulation.

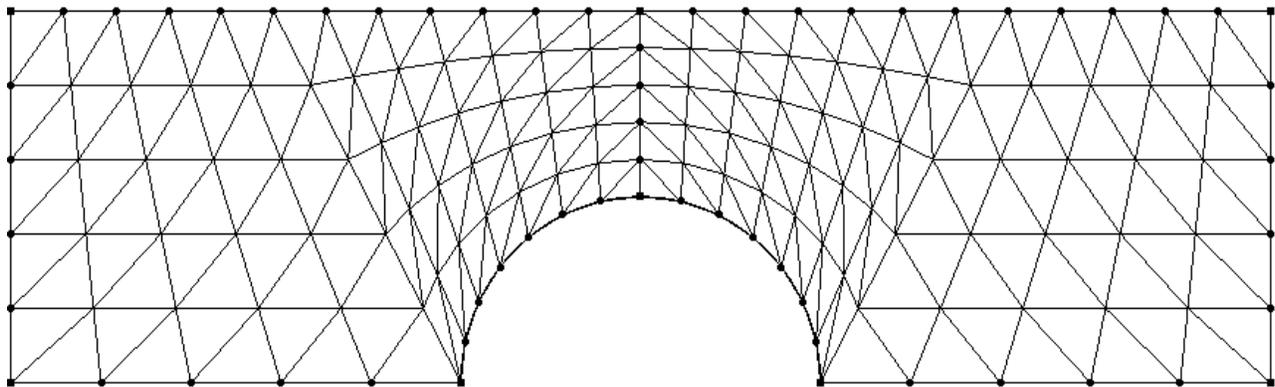


Figure 9. Discretization of domain by 240 triangular elements for stream function formulation.

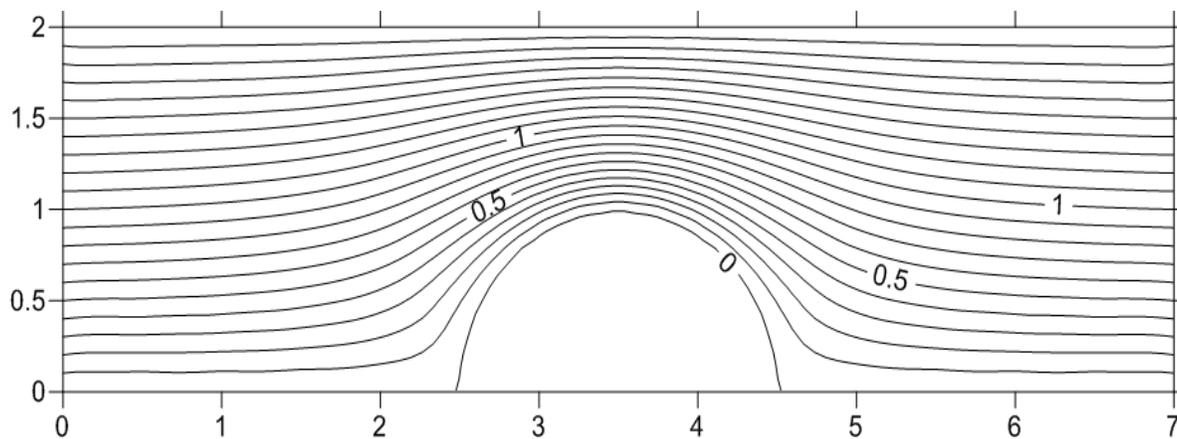


Figure 10. Stream function contours around the half circular cylinder.

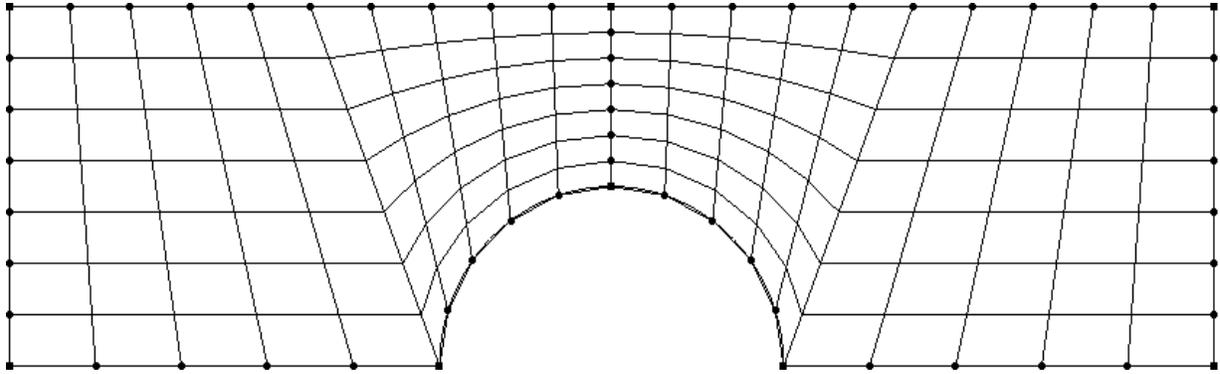


Figure 11. Discretization of computational domain by 140 quadrilateral elements for velocity potential.

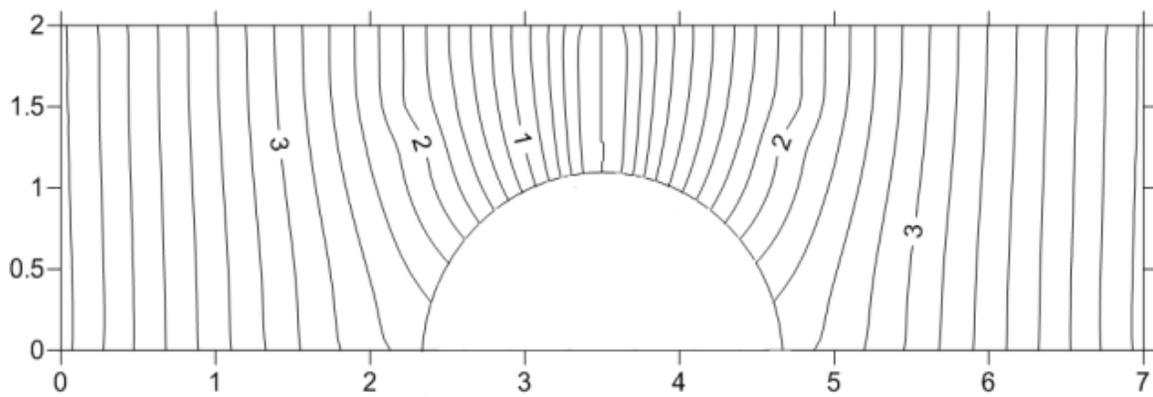


Figure 12. Velocity potential contours around the half circular cylinder.

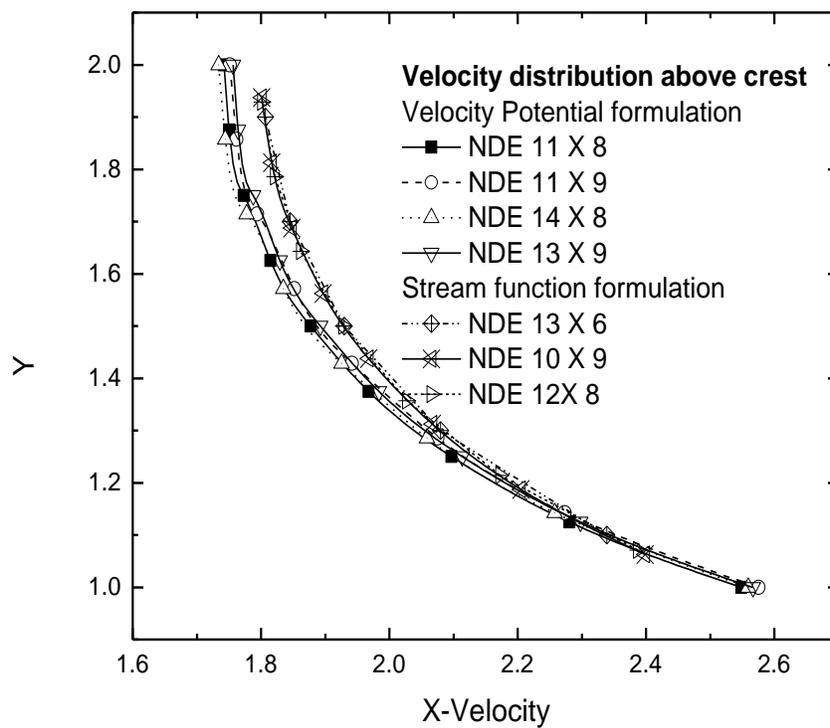


Figure 13. Velocity distributions above the crest of cylinder.

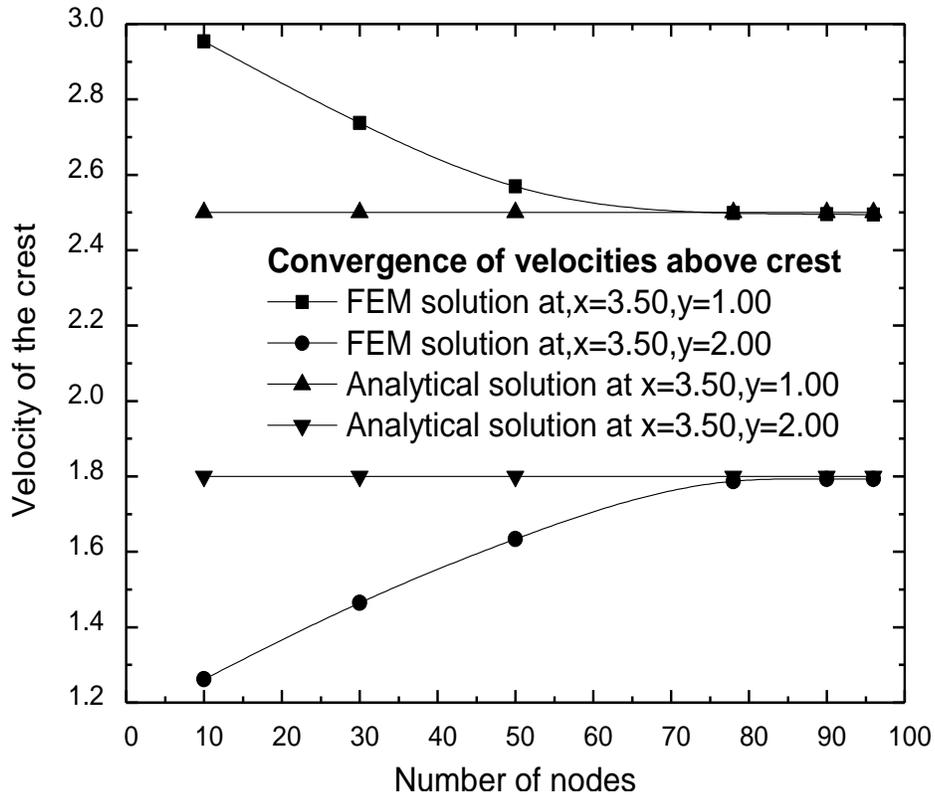


Figure 14. Error analysis for Ψ formulation showing convergence of velocities above the crest.

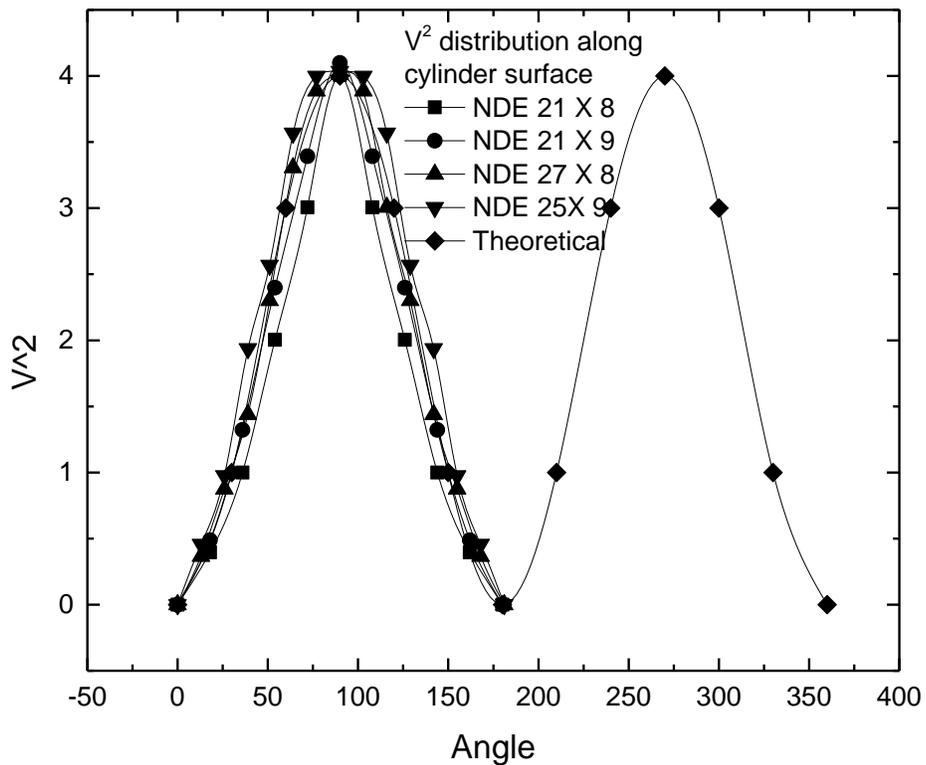


Figure 15. Velocity profile along cylinder surface.

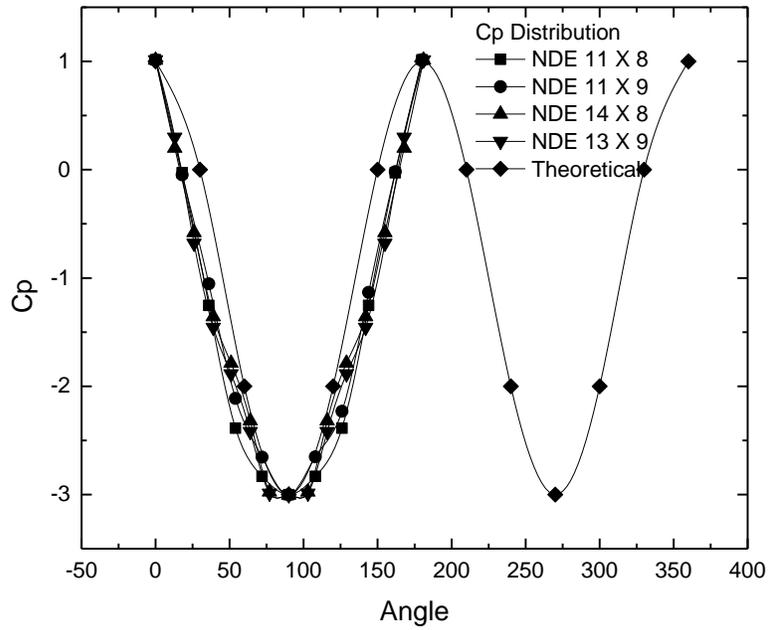


Figure 16. Distribution of pressure coefficient (C_p).

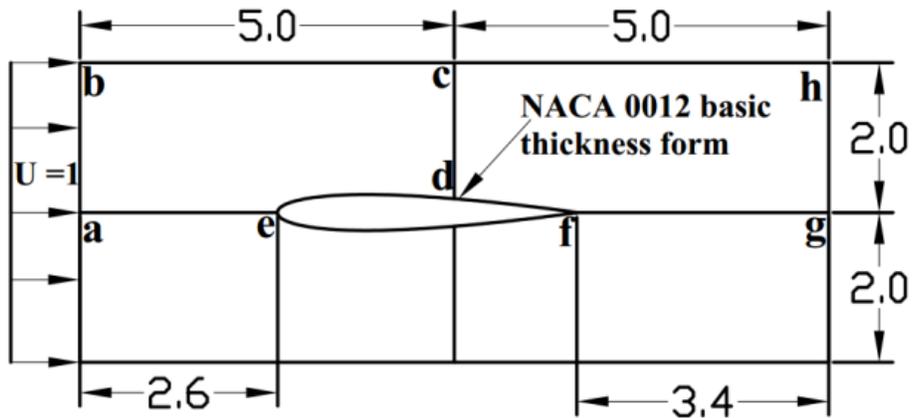


Figure 17. Computational domain for flow around NACA 0012 hydrofoil.

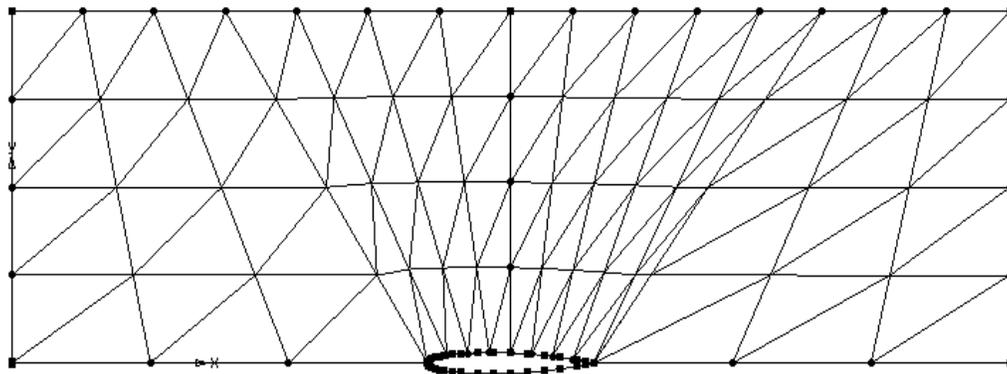


Figure 18. Discretization of computational domain by 120 triangular elements for the hydrofoil.

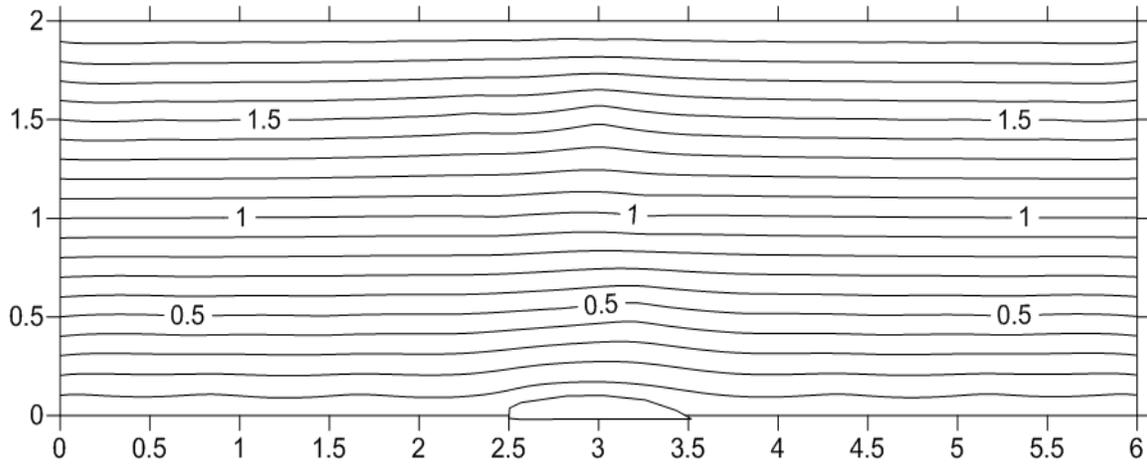


Figure 19. Stream function contours for flow around NACA 0012 hydrofoil.

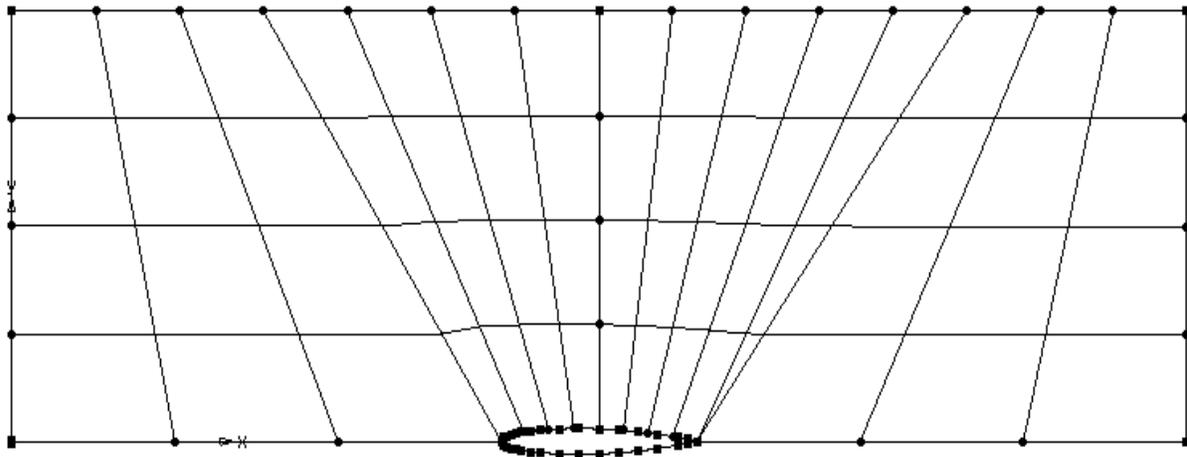


Figure 20. Discretization of computational domain by 60 quadrilateral elements for the hydrofoil.

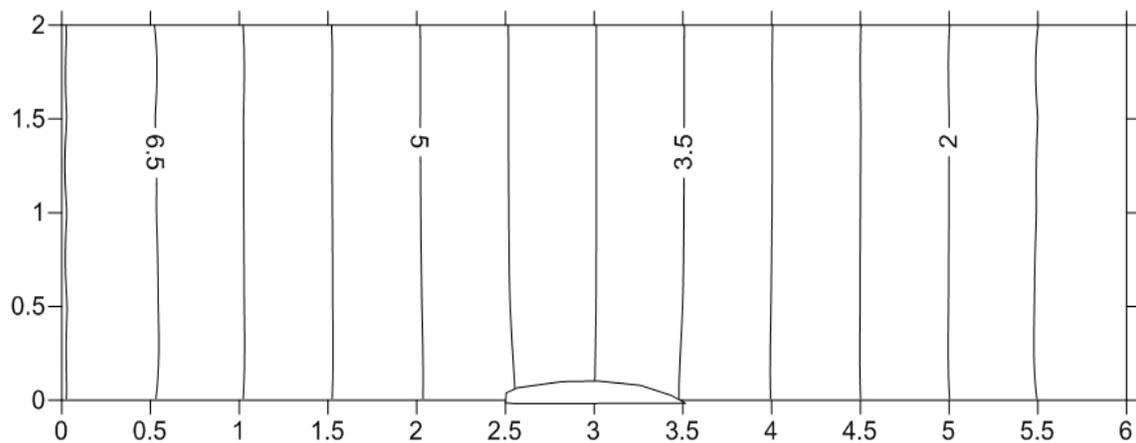


Figure 21. Contours of velocity potential.

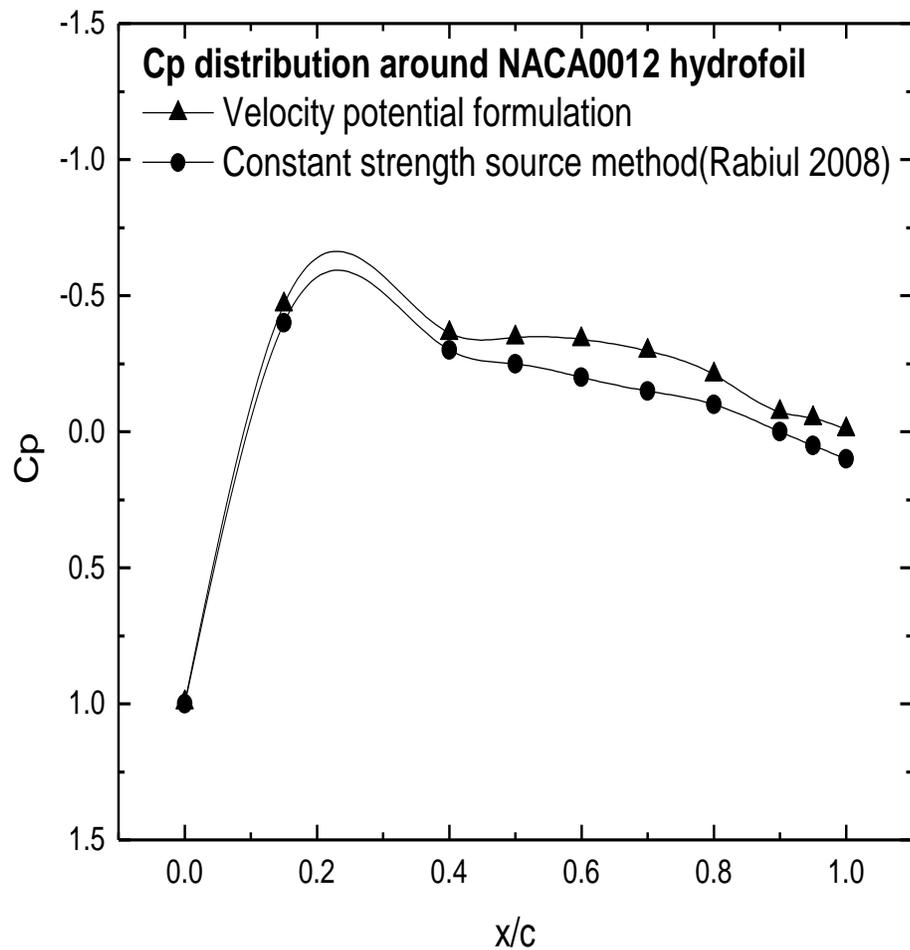


Figure 22. Chord wise pressure variations.

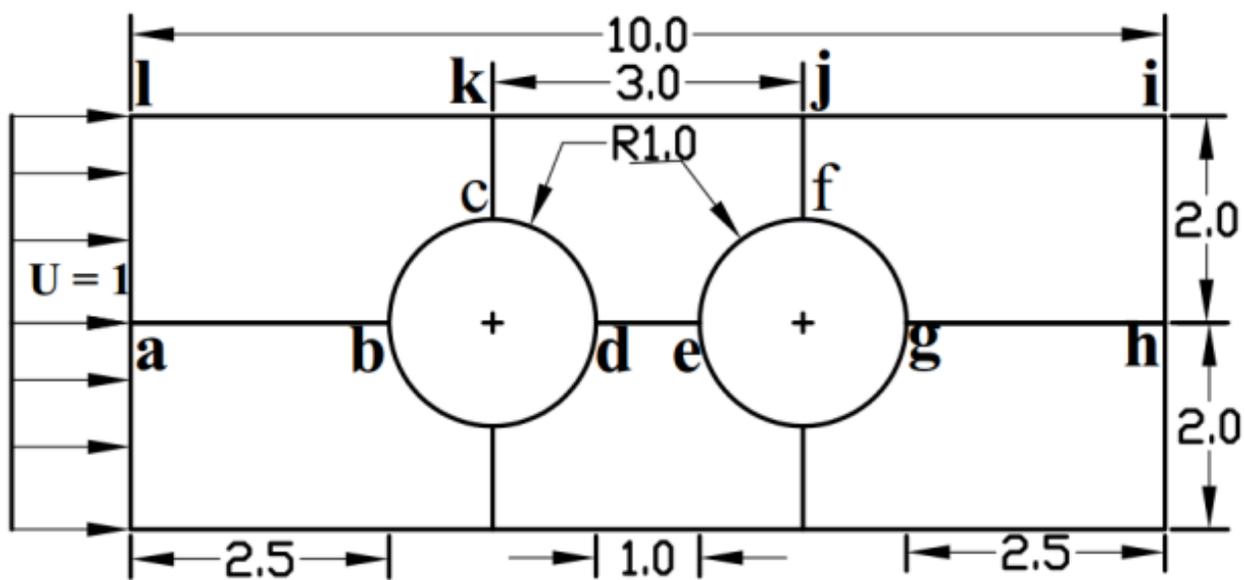


Figure 23. Computational domain for flow around two circular cylinders.

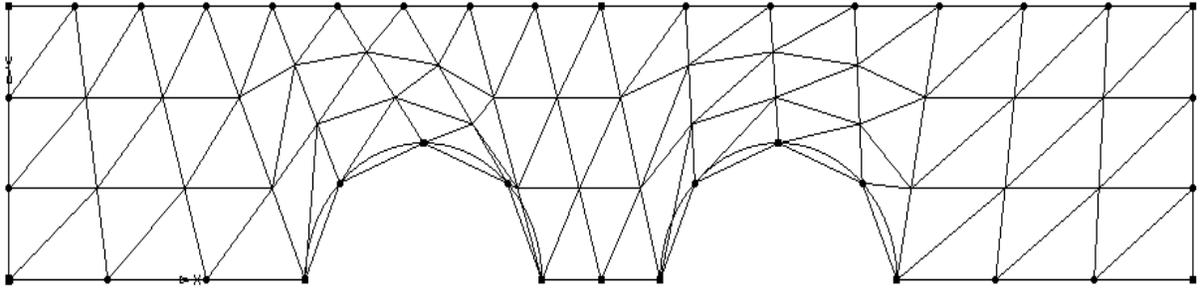


Figure 24. Discretization of domain by 48 triangular elements for flow around two circular cylinders.

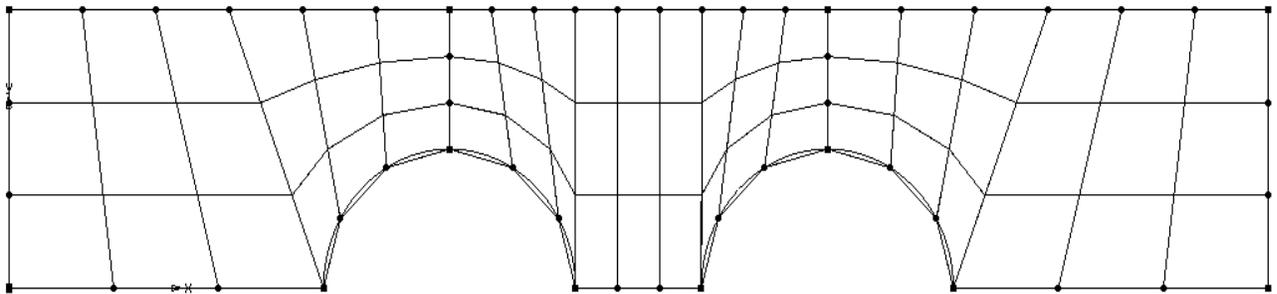


Figure 25. Mesh arrangement for velocity potential formulation.

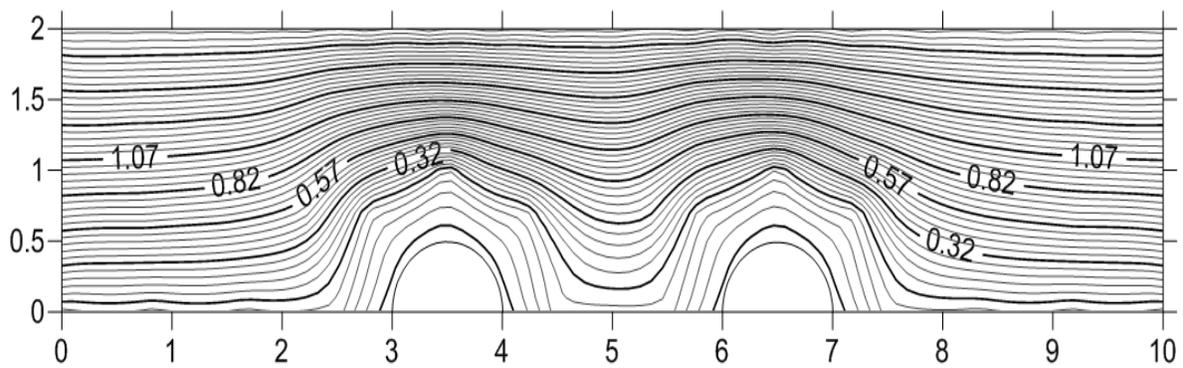


Figure 26. Stream lines contours from stream function formulation.

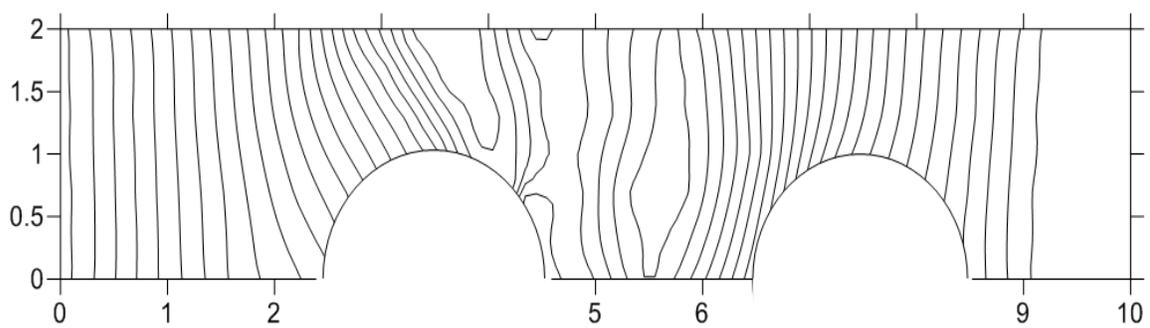


Figure 27. Velocity potential lines.

(iii) The calculated pressure co-efficient shows the pressure decreasing around the forwarded face from the initial total pressure at the stagnation point and reaching a minimum pressure at the top of the cylinder.

(iv) The calculated results depend to a certain extent on the discretization of the computational domain and accuracy increases with increase of number of elements.

Nomenclature: Ψ :Stream function, Φ :Velocity Potential function, N :Shape function, U :Free stream velocity, k^e :Element coefficient matrix, f^e :Element force vector.

Conflict of Interest

The authors have not declared any conflict of interest.

REFERENCES

- Turner MJ, Clough RL, Martin HC, Topp LJ (1956). Stiffness and deflection analysis of complex structures. *J. Aero. Sci.* 23(9):805-825.
- Chung TJ (1977). *Finite Element Analysis in Fluid Dynamics*. McGraw Hill, NY, USA. pp. 68-79.
- Thom A (1933). "The flow past circular cylinders at low speeds". *Proc. Royal Soc.* 141(845):651-669.
- Kawaguti M (1953). Discontinuous flow past a circular cylinder." *J. Phys. Soc. Jap.* 8:403-899.
- Payne RB (1958). Calculation of Unsteady Viscous Flow Past Cylinder. *J. Fluid Mech.* 4:81.
- Oden JT (1969). "A General Theory of Finite Elements". *Int. J. Numer. Methods Eng.* 1:247-259.
- Dennis SCR, Chung GZ (1970). "Numerical Solutions for Steady Flow past a Circular Cylinder at Reynolds Numbers Up to 100". *J. Fluid Mech.* 42(3):471-489.
- Tong P (1971). "The Finite Element Method for Fluid Flow". In: Gallagher RH, Oden JT, Yamada Y (eds.), *Recent Advances in Matrix Method of Structural Analysis and Design*. University of Alabama Press, Alabama. 904pp.
- Olson MD (1974). "Variational-Finite Element Methods for Two-Dimensional and Axisymmetric Navier-Stokes Equations". *Proceedings of the fourth International Symposium on Finite Element Methods in Flow Problems*, Swansea, UK.
- Hafez M (2004). "Inviscid Flows over a Cylinder". *Comput. Methods Appl. Mech. Eng.* 193:1981-1995.
- Tarafder MS, Khalil GM, Islam MR (2010). "Analysis of potential flow around two-dimensional hydrofoil by source based lower and higher order panel method". *J. Institut. Eng. Malaysia* (71):2.