academicJournals

Vol. 9(17), pp. 752-757, 15 September, 2014
DOI: 10.5897/SRE2014.5997
Article Number: 352632D47287
ISSN 1992-2248 © 2014

http://www.academicjournals.org/SRE

Scientific Research and Essays

Full Length Research Paper

Some multistep higher order iterative methods for non-linear equations

Nazir Ahmad Mir[1*] and Naila Rafiq[2]

[1]Department of Basic Sciences, Riphah International University, I-14 Islamabad, Pakistan.
[2]Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, Pakistan.

Received 6 June, 2014; Accepted 20 August, 2014

We establish here new three and four-step iterative methods of convergence order seven and fourteen to find roots of non-linear equations. The new developed iterative methods are variants of Newton's method that use different approximations of first derivatives in terms of previously known function values, thus improving efficiency indices of the methods. The seventh and fourteenth order iterative methods use four and five function values including one derivative of a function. So, the efficiency indices of these methods are $\sqrt[4]{7} = 1.6265 \; and \; \sqrt[5]{14} = 1.6952$, respectively. Numerical examples are given to show the performance of described methods.

Key words: Non-linear equation, Iterative methods, convergence order, efficiency index.

INTRODUCTION

Newton's method is a well known and commonly used quadratically convergent iterative method for finding roots of non-linear equation:

$$f(x) = 0, \tag{1}$$

which is given by

$$x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)} \tag{2}$$

In recent years, researchers have made many modifications in this method to get higher order iterative methods. These methods are developed using various techniques by introducing some more steps to Newton's method. In this way, not only the convergence order but efficiency index of the method may also be increased (Chun, 2007; Mir and Rafiq, 2013; Osada, 1998; Potra and Ptak, 1984; Sharma, 2005).

Recently, the researchers introduced three-step iterative methods of convergence order seven to eight (Kou and Wang, 2007; Mir and Rafiq, 2014; Bi et al., 2009). Most recently, Neta and Petkovic (2010) introduced four-step iterative method of optimal convergence order sixteen for solving non-linear equations. In 2011, Sargolzaei and Soleymani (2011) introduced 4-step method of convergence order fourteen with five function evaluation and thus have efficiency index $\sqrt[5]{14} = 1.6952$. In 1981, Neta introduced 4-step method but did not prove its convergence order. In 2010

*Corresponding author. E-mail: nazirahmad.mir@gmail.com

Mathematics Subject Classification (2010): 65H05, 65B99.

(Geum and Kim, 2010) prove that the order of convergence of Neta's method is fourteen. To see more in this direction, we refer to Neta (1981), Sargolzaei and Soleymani (2011), Soleymani and Sharifi (2011).

Motivated in this direction, we also introduce here three-step and four-step iterative methods of convergence order seven and fourteen with efficiency indices, namely $\sqrt[4]{7} = 1.6265$ *and* $\sqrt[5]{14} = 1.6952$ respectively.

THE ITERATIVE METHODS

Consider two-step optimal convergent order four methods for solution of non-linear equation by Mir et al. (2013):

$$x_{n+1} = x_n - \{\alpha\mu(x_n) + \beta v(x_n)\};\ for\ \alpha = -1, \beta = 2$$
$$x_{n+1} = x_n - \{\alpha\mu(x_n) + \beta\psi(x_n) + \gamma v(x_n)\};$$
$$for\ \alpha = 1, \beta = 1, \gamma = -1$$

Where

$$\mu(x_n) = \frac{f(x_n) + f(y_n)}{f'(x_n)},$$
$$v(x_n) = \frac{f^2(x_n)}{f'(x_n)[f(x_n) - f(y_n)]},$$
$$\psi(x_n) = \frac{f(x_n) + 2f(y_n)}{f(x_n) + f(y_n)} \frac{f(x_n)}{f'(x_n)}$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

At third step, we add Newton's method with an approximation of derivative $f'(z_n)$ from Mir et al. (2014) which is given by:

$$f'(z_n) = f[x_n, z_n] - f[x_n, y_n, z_n](x_n - z_n) \tag{3}$$

where $f[x_n, z_n]$ and $f[x_n, y_n, z_n]$ are the divided differences which are defined by:

$$f[x_n, z_n] = \frac{f(x_n) - f(z_n)}{x_n - z_n}$$

and

$$f[x_n, y_n, z_n] = \frac{f[x_n, y_n] - f[y_n, z_n]}{x_n - z_n}$$

We therefore propose the following three-step algorithms:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ z_n = x_n - \dfrac{f(x_n)}{f'(x_n)}\left(\dfrac{2f(x_n)}{f(x_n) - f(y_n)} - \dfrac{f(x_n) + f(y_n)}{f'(x_n)}\right) \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f[x_n, z_n] - f[x_n, y_n, z_n](x_n - z_n)} \end{cases} \tag{4}$$

and

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} * q \\ where\ q = \left(\dfrac{f(x_n) + f(y_n)}{f'(x_n)} + \dfrac{f(x_n) + 2f(y_n)}{f(x_n) + f(y_n)} - \dfrac{f(x_n)}{f(x_n) - f(y_n)}\right) \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f[x_n, z_n] - f[x_n, y_n, z_n](x_n - z_n)} \end{cases} \tag{5}$$

We can further improve the efficiency indices of Methods (4) and (5) by adding Newton's method as one more step with the approximation of derivative $f'(w_n)$ which is given by

$$f'(w_n) = f[x_n, w_n] + (f[y_n, x_n, z_n] - f[y_n, x_n, w_n]$$
$$-f[z_n, x_n, w_n])(x_n - w_n) \tag{6}$$

from Sargolzaei and Soleymani (2011) and Soleymani and Sharifi (2011) which increase just one function value. Thus, we propose the following four-step algorithms:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}\left(\dfrac{2f(x_n)}{f(x_n) - f(y_n)} - \dfrac{f(x_n) + f(y_n)}{f'(x_n)}\right) \\ w_n = z_n - \dfrac{f(z_n)}{f[x_n, z_n] - f[x_n, y_n, z_n](x_n - z_n)} \\ x_{n+1} = w_n - \dfrac{f(w_n)}{f'(w_n)},\ where\ f'(w_n)\ is\ given\ by\ (6) \end{cases} \tag{7}$$

and

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} * q \\ where\ q = \left(\dfrac{f(x_n) + f(y_n)}{f'(x_n)} + \dfrac{f(x_n) + 2f(y_n)}{f(x_n) + f(y_n)} - \dfrac{f(x_n)}{f(x_n) - f(y_n)}\right) \\ w_n = z_n - \dfrac{f(z_n)}{f[x_n, z_n] - f[x_n, y_n, z_n](x_n - z_n)} \\ x_{n+1} = w_n - \dfrac{f(w_n)}{f'(w_n)},\ where\ f'(w_n)\ is\ given\ by\ (6) \end{cases} \tag{8}$$

Convergence analysis

We now use Maple 10.0 to derive error equations of the iterative methods described by Equations (4), (5), (7) and (8). We prove that the iterative Methods (4) and (5) are of convergence order seven and Methods (7) and (8) are of convergence order fourteen.

Theorem 1: Let $\omega \in I$ be a simple root of a sufficiently differentiable function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ in an open interval I. If x_0 is sufficiently close to ω then the convergence order of three-step method described by (4) is seven and the error equation is given by:

$$e_{n+1} = c_3 c_2 (c_3 c_2 - 3c_2^3) e_n^7 + O(e_n^8).$$

Proof: Let ω be a simple root of f and $x_n = \omega + e_n$. By Taylor's expansion, we have:

$$f(x_n)=f'(\omega)\left(e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+c_5e_n^5+c_6e_n^6\right)+O\left(e_n^7\right), \tag{9}$$

$$f'(x_n)=f'(\omega)\left(1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+6c_6e_n^5\right)+O(e_n^6), \tag{10}$$

where $c_k=\left(\frac{1}{k!}\right)\frac{f^{(k)}(\omega)}{f'(\omega)},\ k=2,3,\ldots$.

Substituting Equations (9) and (10) in Equation (4), we obtain:

$$y_n=\omega+c_2e_n^2+\left(-2c_2^2+2c_3\right)e_n^3-\left(7c_2c_3-4c_2^3-3c_4\right)e_n^4+O\left(e_n^5\right). \tag{11}$$

Thus, using Taylor's series, we have:

$$f(y_n)=f'(\omega)\left(c_2e_n^2+2\left(c_3-c_2^2\right)e_n^3+\left(-7c_2c_3+3c_4+5c_2^3\right)e_n^4+O(e_n^5)\right). \tag{12}$$

Substituting Equations (9), (10), (11) and (12) in Equation (4), we have:

$$z_n=\omega+(-c_2c_3+3c_2^3)e_n^4+(-2c_3^2-18c_2^4-2c_2c_4+20c_3c_2^2)e_n^5+(70c_2^5-130c_3c_2^3$$
$$-7c_4c_3-3c_2c_5+42c_2c_3^2+30c_4c_2^2)e_n^6+O\left(e_n^7\right). \tag{13}$$

Now, we expand $f(z_n)$ at ω using Taylor expansion:

$$\begin{aligned}f(z_n)=f'(\omega)((&-c_2c_3+3c_2^3)e_n^4+(-18c_2^4+20c_3c_2^2-2c_2c_4-2c_3^2)e_n^5\\&+(30c_4c_2^2-7c_4c_3+70c_2^5-3c_2c_5+42c_2c_3^2-130c_3c_2^3)e_n^6\\&+(18c_3c_5-16c_2^3c_4-16c_2^2c_5+12c_2c_6-2c_2^2c_3^2+152c_2^4c_3+10c_4^2\\&-4c_7-52c_2c_4c_3-12c_3^3-92c_2^6)e_n^7)+O\left(e_n^8\right).\end{aligned} \tag{14}$$

Using Taylor series, we have from Equation (4)

$$f'(z_n)=f'(\omega)(1+c_2c_3e_n^3-(6c_2^4-c_2c_4-2c_3^2)e_n^4+O\left(e_n^5\right)). \tag{15}$$

Further substituting (13), (14) and (15) in three-step method (4), we have:

$$x_{n+1}=\omega+c_3c_2(c_3c_2-3c_2^3)e_n^7+O(e_n^8).$$

This implies,

$$e_{n+1}=c_3c_2(c_3c_2-3c_2^3)e_n^7+O(e_n^8).$$

Hence, the theorem is proved.

Theorem 2: Let $\omega\in I$ be a simple root of a sufficiently differentiable function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ in an open interval I. If x_0 is sufficiently close to ω then the convergence order of three-step method described by algorithm (5) is seven and the error equation is given by:

$$e_{n+1}=c_3c_2(c_3c_2-5c_2^3)e_n^7+O(e_n^8).$$

Proof: Similar to Theorem 1.

Theorem 3: Let $\omega\in I$ be a simple root of a sufficiently differentiable function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ in an open interval I. If x_0 is sufficiently close to ω then the convergence order of three-step method described by algorithm (7) has order of convergence fourteen and satisfy the error equation:

$$e_{n+1}=(b_7c_2a_4c_4)e_n^{14}+O(e_n^{15}),$$

where $a_4=-c_2c_3+3c_2^3$ *and* $b_7=-a_4c_2c_3.$

Proof: Let us consider the error Equation (13) as follows:

$$z_n=\omega+a_4e_n^4+a_5e_n^5+a_6e_n^6++a_7e_n^7+O\left(e_n^8\right), \tag{16}$$

Where

$$a_4=-c_2c_3+3c_2^3,$$
$$a_5=20c_3c_2^2-18c_2^4-2c_2c_4-2c_3^2,$$
$$a_6=-130c_3c_2^3-3c_2c_5+30c_4c_2^2+42c_2c_3^2-7c_4c_3+70c_2^5,$$
$$a_7=18c_3c_5-16c_2^3c_4-16c_2^2c_5+12c_2c_6-2c_2^2c_3^2+152c_2^4c_3+10c_4^2-4c_7-52c_2c_4c_3-12c_3^3-92c_2^6$$

and $c_k=\left(\frac{1}{k!}\right)\frac{f^{(k)}(\omega)}{f'(\omega)}$, for $k=2,3,\ldots$.

Now,

$$f(z_n)=f'(\omega)[a_4e_n^4+a_5e_n^5+a_6e_n^6++a_7e_n^7+(a_8+c_2a_4^2)e_n^8+O\left(e_n^9\right)]. \tag{17}$$

From Equation (3):

$$f'(z_n)=f'(\omega)(1-c_2c_3e_n^3+(2c_3c_2^2-c_2c_4-2c_3^2+2c_2a_4)e_n^4++O\left(e_n^5\right)).$$

Thus,

$$\frac{f(z_n)}{f'(z_n)}=a_4e_n^4+a_5e_n^5+a_6e_n^6+(a_7+c_2c_3a_4)e_n^7+O\left(e_n^8\right) \tag{18}$$

Now, substituting Equations (16) and (18) in Equation (7), we get:

$$\begin{aligned}w_n=\omega&-a_4c_2c_3e_n^7+(-a_4c_2c_4+2a_4c_3c_2^2-a_5c_2c_3-2a_4c_3^2+c_2a_4^2)e_n^8\\&+(2c_2^2c_3a_5-2c_3^2a_5-c_2c_3a_6-c_2c_4a_5)e_n^9+\ldots+O\left(e_n^{15}\right).\end{aligned} \tag{19}$$

We rewrite the error Equation (19) as follows:

$$w_n=\omega+b_7e_n^7+b_8e_n^8+b_9e_n^9+\ldots+O\left(e_n^{15}\right). \tag{20}$$

where $b_7=-a_4c_2c_3,$
$b_8=-2a_4c_3^2+2a_4c_3c_2^2-a_5c_2c_3-a_4c_2c_4+c_2a_4^2$ and
$b_9=-2c_3^2a_5-c_2c_3a_6+2c_2^2c_3a_5-c_2c_4a_5$

Now, by Taylor expansion,

$$f(w_n)=f'(\omega)[b_7e_n^7+b_8e_n^8+b_9e_n^9+\ldots+O\left(e_n^{14}\right)]. \tag{21}$$

Substituting Equations (9-12), (16), (17), (20) and (21) in Equation (6), we have

$$f'(w_n) = f'(\omega)\left(1 + (c_2 a_4 c_4 + c_2 b_7)e_n^7 + O\left(e_n^8\right)\right). \qquad (22)$$

Substituting Equations (20), (21) and (22) we have the following error equation for the four-step method (7):

$$x_{n+1} = \omega + (b_7 c_2 a_4 c_4)e_n^{14} + O\left(e_n^{15}\right).$$

This implies $e_{n+1} = (b_7 c_2 a_4 c_4)e_n^{14} + O(e_n^{15})$.

Theorem 4: Let $\omega \in I$ be a simple root of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ in an open interval I. If x_0 is sufficiently close to ω then the convergence order of three-step method described by algorithm (8) has order of convergence fourteen:

$$e_{n+1} = (b_7 c_2 a_4 c_4)e_n^{14} + O(e_n^{15}),$$

$$where\ b_7 = c_3 c_2 (c_3 c_2 - 5c_2^3)\ and\ a_4 = -c_2 c_3 + 5c_2^3.$$

Proof: Similar to Theorem 3.

NUMERICAL EXAMPLES

Comparison of 7th order convergent methods

Now, we compare our seventh order convergent methods, namely $MN_7 - 1$ and $MN_7 - 2$ described by (4) and (5) respectively, with second order convergent Newton's method NW (11) sixth order convergent method (G6) Grau and Diaz-Barrero (2006) and seventh order convergent method (G7) Kou et al. (2007). The result of the numerical comparison of various methods is shown in Table 3 on the same number of function evaluations (TNFE=12), that is, on the third iteration with 350 significant digits and convergence criterion as follows:

$$|f(x_n)| < 10^{-300}.$$

Examples are taken from (Bi et al., 2009) which are used for comparison as follows (Table 1):

Comparison of 14th order methods

We compare our methods namely $MN_{14} - 1$ and $MN_{14} - 2$ described by (7) and (8) with Neta's fourteenth order method (N14) (1981) at the same number of function evaluations TNFE=12. The absolute values of the given test functions at first three iterations are given in Table 4. The computation is carried out with 2500 significant digits and with the following stopping criteria:

$$|f(x_n)| < \epsilon, \quad \text{where } \epsilon = 10^{-2450}$$

Table 1. Examples for 7th order methods.

Example	Roots
$f_1(x) = x^5 + x^4 + 4x^2 - 15$	1.347428098968305
$f_2(x) = \sin x - x/3$	2.278862660075828
$f_3(x) = 10xe^{-x^2} - 1$	1.679630610428450
$f_4(x) = \cos x - x$	0.7390851332151606
$f_5(x) = e^{-x^2+x+2} - 1$	1.00000000000000
$f_6(x) = e^{-x} + \cos x$	1.746139530408012
$f_7(x) = \ln(x^2 + x + 2) - x + 1$	4.152590736757158
$f_8(x) = \sin^{-1}(x^2 - 1) - \frac{1}{2}x + 1$	0.5948109683983692

Table 2. Examples for 14th order methods.

Example	Roots
$f_1(x) = e^x + x - 20,$	$\omega \approx 2.8424389537844470...,$
$f_2(x) = \sqrt{x^2 + 2x + 5} - 2\sin x - x^2 + 3$	$\omega \approx 2.3319676558839640...,$
$f_3(x) = 2x\cos x + x - 3,$	$\omega \approx -3.0346643069740450...,$
$f_4(x) = (x-1)^6 - 1$	$\omega = 2$

Here, we use the following test functions (Parviz and Soleymani, 2011) for comparison (Table 2). We observe that the numerical results are comparable or better in some cases.

Conclusion

In this article, we modified the existing methods of Mir et al. (2014) with the introduction of one and two steps more in such a way that the modified methods have improved convergence order that is, from fourth order to seventh order and then to fourteenth order as well as with their efficiencies improved. Modified methods are comparable and have better results as compared to the existing methods shown in Tables 3 and 4. The three-step methods are of seventh order convergent with four function evaluations per iteration and thus having computational efficiency $\sqrt[4]{7} = 1.6265$. The four-step methods are of fourteenth order convergent methods with five function evaluations per iteration and thus having computational efficiency $\sqrt[5]{14} = 1.6952$.

Conflict of Interest

The authors have not declared any conflict of interest.

Table 3. TNFE=Total number of function evaluation.

Comparison of 7th order methods on the same TNFE=12					
Example	**Methods**				
	NW	**G6**	**G7**	**MN_7-1**	**MN_7-2**
$f_1, x_0 = 1.6$	3.6e-39	8.2e-140	1.62e-216	1.3e-233	4.7e-321
$f_2, x_0 = 2.0$	4.20e-57	5.3e-166	1.14e-244	2.3e-315	2.3e-251
$f_3, x_0 = 1.8$	1.22e-57	9.3e-187	1.34e-281	4.7e-297	9.0e-264
$f_4, x_0 = 1.0$	3.00e-83	4.1e-237	0.0e+00	0.0e+00	0.0e+00
$f_5, x_0 = -0.5$	1.04e-26	3.28e-79	8.94e-118	1.9e-145	8.7e-135
$f_6, x_0 = 2.0$	9.24e-85	1.5e-233	1.29e-338	3.0e-350	3.0e-350
$f_7, x_0 = 3.2$	2.81e-74	1.0e-207	2.88e-312	1.0e-349	3.8e-343
$f_8, x_0 = 1.0$	1.78e-54	1.0e-165	1.30e-215	3.2e-269	1.6e-241

Table 4. TNFE=Total number of function evaluation.

Comparison of 14^{th} order methods on the same TNFE=12				
Example	$\lvert f(x_n) \rvert$	**Methods**		
		N_{14}	$MN_{14}-1$	$MN_{14}-2$
$f_1, x_0 = 3.5$	$\lvert f_1(x_1) \rvert$	0.2e-6	1.0e-7	1.8e-7
	$\lvert f_1(x_2) \rvert$	0.4e-113	9.3e-119	4.4e-115
	$\lvert f_1(x_3) \rvert$	0.1e-1608	1.1e-1673	1.2e-1621
$f_2, x_0 = 0.5$	$\lvert f_2(x_1) \rvert$	0.1e-7	1.2e-8	2.5e-8
	$\lvert f_2(x_2) \rvert$	0.6e-124	1.5e-125	1.6e-121
	$\lvert f_2(x_3) \rvert$	0.3e-1751	2.8e-1762	4.9e-1760
$f_3, x_0 = -3.2$	$\lvert f_3(x_1) \rvert$	0.5e-3	4.6e-3	2.2e-1
	$\lvert f_3(x_2) \rvert$	0.3e-44	1.7e-33	8.0e-11
	$\lvert f_3(x_3) \rvert$	0.7e-621	1.5e-459	4.2e-142
$f_4, x_0 = 2.6$	$\lvert f_4(x_1) \rvert$	0.1	2.7e-2	5.0e-2
	$\lvert f_4(x_2) \rvert$	0.3e-16	1.6e-27	1.2e-23
	$\lvert f_4(x_3) \rvert$	0.3e-235	1.5e-380	5.9e-326

REFERENCES

Bi W, Ren H, Wu Q (2009). Three-step iterative methods with eighth order convergence for solving non-linear equations. J. Comput. Appl. Math. 225:105-112. http://dx.doi.org/10.1016/j.cam.2008.07.004

Chun C (2007). Some third-order families of iterative methods, methods for solving nonlinear equation, Appl. Math. Comput. 188:924-933. http://dx.doi.org/10.1016/j.amc.2006.10.043

Grau M, Diaz-Barrero JL (2006). An improvement to Ostrowski root-finding method. Appl. Math. Comput. 173: 450-456. http://dx.doi.org/10.1016/j.amc.2005.04.043

Kou J, Li Y, Wang X (2007).Some variants of Ostrowski's method with seventh-order convergence, J. Comput. Appl. Math. 209:153-159. http://dx.doi.org/10.1016/j.cam.2006.10.073

Mir NA, Rafiq N (2013). Some new efficient Iterative method for solving non-linear equations. WASJ 27(12):1664-1668.

Mir NA, Rafiq N (2014). Some families of three-step optimal convergent order iterative methods for solving non-linear equations. Accepted, Abstr. Anal. Appl.

Neta B (1981). On a family of multi-points methods for non-linear equations. Int. J. Comput. Math. 9:353-361. http://dx.doi.org/10.1080/00207168108803257

Neta B, Petkovic MS (2010). Construction of optimal order nonlinear solvers using inverse interpolation. Appl. Math. Comput. 217(6):2448–2455. http://dx.doi.org/10.1016/j.amc.2010.07.045

Osada N (1998). Improving the order of convergence of iterative functions. J. Comput. Appl. Math. 98:311-315. http://dx.doi.org/10.1016/S0377-0427(98)00131-9

Potra FA, Pták V (1984). Non discrete Induction and Iterative Processes, Research Notes in Mathematics, 103:, Pitman, Boston, Sargolzaei P, Soleymani F (2011). Accuarte fourteenth-order method for solving non-linear equations. Numer. Algor. 58:513-527. http://dx.doi.org/10.1007/s11075-011-9467-4

Sharma JR (2005). A Composite third order Newton-Steffensen method for solving nonlinear equations. Appl. Math. Comput.169:242-246. http://dx.doi.org/10.1016/j.amc.2004.10.040

Soleymani F, Sharifi M (2011). On a class of fifteenth-order iterative formulas for simple roots. Int. Elect. J. Pure Appl. Math. 3(3):245-252.