Scientific Research and Essays Vol. 8(1), pp. 39-42, 4 January, 2013
Available online at http://www.academicjournals.org/SRE
DOI: 10.5897/SRE12.595
ISSN 1992-2248 ©2013 Academic Journals

*Full Length Research Paper*

# Natural partial order on a semi group of self-$E$-preserving transformations

**Chaiwat Namnak* and Ekkachai Laysirikul**

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.

Accepted 24 September, 2012

**Let $T(X)$ denote the full transformation semigroup on a set $X$. For an arbitrary equivalence relation $E$ on $X$, we consider a sub semi-group of $T(X)$ defined by: $T_{SE}(X) = \{\alpha \in T(X): (x, x\alpha) \in E \text{ for each } x \in X\}$ which is called *the self-$E$-preserving transformation semi-group on* $X$. In this paper, we discussed a natural partial order on $T_{SE}(X)$ and characterize when two elements of $T_{SE}(X)$ are related under this order. We also described the left compatibility and right compatibility of each element of $T_{SE}(X)$ with respect to this order.**

**Key words:** Natural partial order, transformation semigroup, compatible.

## INTRODUCTION

For any semi-group $S$, Mitsch (1986) defined the natural partial order on $S$ as follows: for $a, b \in S$, $a \leq b \Leftrightarrow a = xb = by, a = ay$ for some $x, y \in S^1$. This order coincides with the natural partial order for a regular semi-group which is the following: for $a, b \in S$, $a \leq b \Leftrightarrow a = eb = bf$ for some $e, f \in E(S)$, where $E(S)$ is the set of all idempotents of $S$.

Let $X$ be a nonempty set and $T(X)$ be the semigroup under composition of all the full transformations on $X$. Kowol and Mitsch (1986) endowed $T(X)$ with the natural partial order and determined when two elements of $T(X)$ are related under this order in terms of images and kernels. For an equivalence relation $E$ on $X$, let $T_E(X) = \{\alpha \in T(X): \forall (x, y) \in E, (x\alpha, y\alpha) \in E\}$, then $T_E(X)$ becomes a sub semi-group of $T(X)$. Sun et al. (2008) described the natural partial order on $T_E(X)$ and found out elements of $T_E(X)$ which are compatible with respect to this order. In addition, we consider a sub semi-group of $T(X)$ defined as follows: $T_{SE}(X) = \{\alpha \in T(X): \forall x \in X, (x, x\alpha) \in E\}$, which is called *the self-$E$-preserving transformation semi-group on $X$*.

In this paper, we study the natural partial order on $T_{SE}(X)$ and characterize when two elements are related under this order. Furthermore, we determine the left compatible and right compatible elements of $T_{SE}(X)$ with respect to this order.

For $\alpha \in T(X)$, let $\pi(\alpha) = \{ y\alpha^{-1}: y \in X\alpha\}$. Hence, $\pi(\alpha)$ is a partition of $X$.

Moreover, we define a mapping $\alpha_*: \pi(\alpha) \to X\alpha$ as in Ma et al. (2010) corresponding to $\alpha$ by $P\alpha_* = x\alpha$ for all $P \in \pi(\alpha)$ and $x \in P$.

Then $\alpha_*$ is a bijection from $\pi(\alpha)$ onto $X\alpha$. For each $A \in X/E$, we define

$$\pi_A(\alpha) = \{ P \in \pi(\alpha): P \cap A \neq \emptyset\}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of $X$. We say that $\mathcal{B}$ is a *refinement* of $\mathcal{A}$ or $\mathcal{B}$ *refines* $\mathcal{A}$ if $\cup \mathcal{B} = \cup \mathcal{A}$ and for every $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $B \subseteq A$.

*Corresponding author. E-mail: chaiwatn@nu.ac.th. Tel: +668-1-374-2008. Fax: 055-261025.

Throughout of this paper, let $X$ be a nonempty set and $E$ an arbitrary equivalence relation on $X$. Next, we introduce useful propositions.

**Proposition 1.1.** Let $\alpha \in T_{SE}(X)$. If $P \in \pi(\alpha)$, then there exists $A \in X/E$ such that $P \subseteq A$. Hence $\pi(\alpha)$ refines $X/E$.

***Proof.*** Let $P \in \pi(\alpha)$ and $x \in P$. Then there exists $A \in X/E$ such that $x \in A$. Let $p \in P$. Since $\alpha \in T_{SE}(X)$, we have $(p, p\alpha)$, $(x, x\alpha) \in E$. Since $p\alpha = x\alpha$, by transitive of $E$ we deduce that $(p, x) \in E$. Therefore $p \in A$, hence $P \subseteq A$. ■

**Proposition 1.2.** Let $\alpha \in T_{SE}(X)$ and $A \in X/E$. Then the following statements hold:

1. $A = \bigcup \pi_A(\alpha)$.
2. $A\alpha \subseteq A$.

***Proof.***
1. For each $P \in \pi_A(\alpha)$, we have $P \in \pi(\alpha)$ and $P \cap A \neq \emptyset$. By Proposition 1.1, there exists $B \in X/E$ such that $P \subseteq B$. Since $A, B \in X/E$ and $\emptyset \neq A \cap B$, we have $A = B$. This proves that $\bigcup \pi_A(\alpha) \subseteq A$. Next, let $x \in A$. Since $\pi(\alpha)$ is a partition of $X$, there exists some $P \in \pi(\alpha)$ such that $x \in P$, so $P \cap A \neq \emptyset$, hence $P \in \pi_A(\alpha)$. Thus $x \in \bigcup \pi_A(\alpha)$, hence $A \subseteq \bigcup \pi_A(\alpha)$.
2. Let $A \in X/E$ and $x \in A$. Since $(x, x\alpha) \in E$ and $x \in A$, we have $x\alpha \in A$. Hence $A\alpha \subseteq A$. ■

## MAIN RESULTS

First we show that every element of $T_{SE}(X)$ is regular.

**Proposition 2.1.** For $\alpha \in T_{SE}(X)$, $\alpha$ is a regular element. Hence $T_{SE}(X)$ is a regular semi-group.

***Proof.*** Let $\alpha \in T_{SE}(X)$. For each $x \in X\alpha$, we choose and fix an element $x' \in X$ such that $x'\alpha = x$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x' & \text{if } x \in X\alpha; \\ x & \text{otherwise.} \end{cases}$$

Let $x \in X$. If $x \notin X\alpha$, then $(x, x\beta) = (x, x) \in E$. If $x \in X\alpha$, then $(x, x\beta) = (x, x') = (x'\alpha, x') \in E$ since $\alpha \in T_{SE}(X)$. Thus $\beta \in T_{SE}(X)$. For $x \in X$,
$x\alpha\beta\alpha = (x\alpha)\beta\alpha = (x\alpha)'\alpha = x\alpha$.

This proves that $\alpha$ is a regular element of $T_{SE}(X)$. ■
Since $T_{SE}(X)$ is a regular semigroup, as was mentioned we deduce the natural partial order on $T_{SE}(X)$ as follows:

for $\alpha, \beta \in T_{SE}(X)$,

$\alpha \leq \beta \Leftrightarrow \alpha = \delta\beta = \beta\gamma$ for some $\delta, \gamma \in E(T_{SE}(X))$.

The following theorem investigates the condition when $\alpha \leq \beta$ for all $\alpha, \beta \in T_{SE}(X)$.

**Theorem 2.2.** Let $\alpha, \beta \in T_{SE}(X)$. Then $\alpha \leq \beta$ if and only if $\pi(\beta)$ refines $\pi(\alpha)$ and for every $P \in \pi(\alpha), P\alpha_* \in P\beta$.

***Proof.*** Suppose that $\alpha \leq \beta$. Then there exist $\delta, \gamma \in E(T_{SE}(X))$ such that $\alpha = \delta\beta = \beta\gamma$. Let $P \in \pi(\beta)$. Then $P\beta_* = x$ for some $x \in X\beta$. Since $P \subseteq (x\gamma)\alpha^{-1} \in \pi(\alpha)$. This proves that $\pi(\beta)$ is a refinement of $\pi(\alpha)$.

Let $P \in \pi(\alpha)$ and $x \in P$. Then $x\alpha = P\alpha_*$. Since $\delta \in E(T_{SE}(X))$, $x\delta^2 = x\delta$. Therefore $x\alpha = x\delta\beta = (x\delta)\delta\beta = x\delta\alpha$ which implies that $x\delta \in P$. Hence $P\alpha_* = x\delta\beta \in P\beta$.

Conversely, suppose that $\pi(\beta)$ refines $\pi(\alpha)$ and for every $P \in \pi(\alpha), P\alpha_* \in P\beta$. For each $x \in X\beta$, there exists a unique $Q_x \in \pi(\beta)$ such that $x = Q_x\beta_*$. By assumption, there exists a unique $P_x \in \pi(\alpha)$ such that $Q_x \subseteq P_x$. It follows by Proposition 1.1 that $P_x \subseteq A$ for some $A \in X/E$, hence $Q_x \subseteq A$. By Proposition 1.2(2), we have $Q_x\beta_* \in A\beta \subseteq A$ and $P_x\alpha_* \in A\alpha \subseteq A$, hence $(x, P_x\alpha_*) = (Q_x\beta_*, P_x\beta_*) \in E$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} P_x\alpha_* & \text{if } x \in X\beta; \\ x & \text{otherwise.} \end{cases}$$

It is clear that $\gamma \in T_{SE}(X)$. To show that $\gamma \in E(T_{SE}(X))$, let $x \in X$. If $x \notin X\beta$, then $x\gamma^2 = x\gamma$. If $x \in X\beta$, then $x = Q_x\beta_*$, $x\gamma = P_x\alpha_*$ and $Q_x \subseteq P_x$ for some $Q_x \in \pi(\beta)$, $P_x \in \pi(\alpha)$. By assumption, $P_x\alpha_* \in P_x\beta$, there exists $y \in P_x$ such that $P_x\alpha_* = y\beta$. Since $y\beta \in X\beta$, there exists $Q_{y\beta} \in \pi(\beta)$ such that $y\beta = Q_{y\beta}\beta_*$. Hence, $Q_{y\beta} \cap P_x \neq \emptyset$ which implies that $Q_{y\beta} \subseteq P_x$. By definition of $\gamma$, we have $y\beta\gamma = P_x\alpha_*$. Thus, $x\gamma^2 = (P_x\alpha_*)\gamma = y\beta\gamma = P_x\alpha_* = x\gamma$. This shows that $\gamma \in E(T_{SE}(X))$ as required.

To show that $\beta\gamma = \alpha$, let $x \in X$. Then $x\beta = Q_{x\beta}\beta_*$ and $x \in Q_{x\beta} \subseteq P_{x\beta}$ for some $Q_{x\beta} \in \pi(\beta)$ and $P_{x\beta} \in \pi(\alpha)$. Then $x\beta\gamma = P_{x\beta}\alpha_* = x\alpha$, so $\beta\gamma = \alpha$.

Next, for each $P \in \pi(\alpha)$, by assumption $P\alpha_* \in P\beta$, we choose and fix $p' \in P$ such that $P\alpha_* = p'\beta$. Define $\delta : X \to X$ by

$x\delta = p'$ for all $P \in \pi(\alpha)$ and $x \in P$.

Let $x \in X$. Then $x \in P_x$ for some $P_x \in \pi(\alpha)$. By Proposition 1.1, there exists $A \in X/E$ such that $P_x \subseteq A$. Since $x\delta = p'_x \in P_x \subseteq A$, $(x, x\delta) \in E$. Thus $\delta \in T_{SE}(X)$. Consider $x\delta\beta = p'_x\beta = P_x\alpha_* = x\alpha$. Since $p'_x \in P_x$, by definition of $\delta$ we have $p'_x\delta = p'_x$ and $x\delta^2 = p'_x\delta = p'_x = x\delta$.

Therefore $\alpha = \delta\beta$ and $\delta \in E(T_{SE}(X))$, respectively. Thus the theorem is completely proved. ■

**Corollary 2.3.** Let $\alpha, \beta \in T_{SE}(X)$. Then $\alpha \leq \beta$ if and only if for every $A \in X/E$, $\pi_A(\beta)$ is a refinement of $\pi_A(\alpha)$ and for every $P \in \pi_A(\alpha), P\alpha_* \in P\beta$.

***Proof.*** Suppose that $\alpha \leq \beta$. Let $A \in X/E$ and $P \in \pi_A(\beta)$. We then have $P \in \pi(\beta)$ and $P \cap A \neq \emptyset$. By Theorem 2.2, $\pi(\beta)$ is a refinement of $\pi(\alpha)$, there exists $Q \in \pi(\alpha)$ such that $P \subseteq Q$. Thus $\emptyset \neq P \cap A \subseteq Q \cap A$ which implies that $Q \in \pi_A(\alpha)$ and $P \subseteq Q$. Hence $\pi_A(\beta)$ refines $\pi_A(\alpha)$. Moreover, for any $P \in \pi_A(\alpha)$, we then have $P \in \pi(\alpha)$. By Theorem 2.2, $P\alpha_* \in P\beta$.

Conversely, suppose that for every $A \in X/E$, $\pi_A(\beta)$ is a refinement of $\pi_A(\alpha)$ and for every $P \in \pi_A(\alpha)$, $P\alpha_* \in P\beta$. To show that $\alpha \leq \beta$, let $P \in \pi(\beta)$. From Proposition 1.1, $\pi(\beta)$ refines $X/E$. Then there exists $A \in X/E$ such that $P \subseteq A$, so $P \in \pi_A(\beta)$. By assumption, there exists $Q \in \pi_A(\alpha)$ such that $P \subseteq Q$. Since $\pi_A(\alpha) \subseteq \pi(\alpha)$, $\pi(\beta)$ refines $\pi(\alpha)$. Next, let $P \in \pi(\alpha)$. By Proposition 1.1, we have $P \subseteq A$ for some $A \in X/E$, hence $P \in \pi_A(\alpha)$. By assumption, $P\alpha_* \in P\beta$. It follows by Theorem 2.2, $\alpha \leq \beta$ as desired. ■

Recall that for any partial order $\rho$ on a semi-group $S$, an element $c \in S$ is said to be *left compatible* with $\rho$ if for every $(a, b) \in \rho$ implies that $(ca, cb) \in \rho$. Right compatible with $\rho$ is defined dually. Next, we describe the left and right compatible elements in $T_{SE}(X)$.

**Theorem 2.4.** Let $\alpha \in T_{SE}(X)$. Then $\alpha$ is left compatible with $\leq$ on $T_{SE}(X)$ if and only if $\alpha$ is surjective.

***Proof.*** Suppose that $\alpha$ is not surjective. Let $a' \in X \setminus X\alpha$. Then there exists $A \in X/E$ such that $a' \in A$. We choose and fix an element $a \in A\alpha$, hence $a \neq a'$. By Proposition 1.2(2), we have that $a, a' \in A$. Define $\beta: X \to X$ by

$$x\beta = \begin{cases} a' & \text{if } x = a; \\ x & \text{otherwise.} \end{cases}$$

Since $a, a' \in A$, $\beta \in T_{SE}(X)$. We note that $\pi(\beta) = \{\{a, a'\}\} \cup \{\{x\} : x \in X \setminus \{a, a'\}\}$. It is easy to see that $\pi(I_X)$ refines $\pi(\beta)$ and $P\beta_* \in PI_X$ for all $P \in \pi(\beta)$ where $I_X$ is the identity map on $X$. By Theorem 2.2, we deduce that $\beta \leq I_X$. Since $a' \in X\alpha\beta$, we have $Q = a'(\alpha\beta)^{-1} \in \pi(\alpha\beta)$. Then

$$Q = a'(\alpha\beta)^{-1} = a'\beta^{-1}\alpha^{-1} = \{a, a'\}\alpha^{-1} = a\alpha^{-1}.$$

Since $Q\alpha I_X = (a\alpha^{-1})\alpha I_X = \{aI_X\} = \{a\}$, $Q(\alpha\beta)_* = a' \notin Q\alpha I_X$. By Theorem 2.2, we conclude that $\alpha\beta \nleq \alpha I_X$. This proves that $\alpha$ is not left compatible with $\leq$ on $T_{SE}(X)$.

Conversely, assume that $\alpha$ is surjective. Then $Y\alpha^{-1}\alpha = Y$ for all $Y \subseteq X$. Let $\beta, \gamma \in T_{SE}(X)$ be such that $\beta \leq \gamma$. To show that $\alpha\beta \leq \alpha\gamma$ via Theorem 2.2, let $P \in \pi(\alpha\gamma)$. Then $P(\alpha\gamma)_* = y$ for some $y \in X\alpha\gamma$. Since $X\alpha\gamma \subseteq X\gamma$, $y \in X\gamma$. Let $Q \in \pi(\gamma)$ be such that $Q\gamma_* = y$. Since $\beta \leq \gamma$, $Q \subseteq \tilde{P}$ for some $\tilde{P} \in \pi(\beta)$. Since

$$P\alpha\beta = y(\alpha\gamma)^{-1}\alpha\beta = (y\gamma^{-1})\alpha^{-1}\alpha\beta = (y\gamma^{-1})\beta = Q\beta \subseteq \tilde{P}\beta = \{\tilde{P}\beta_*\},$$

we have that $P \subseteq \tilde{P}\beta_*(\alpha\beta)^{-1} \in \pi(\alpha\beta)$.

Next, let $P \in \pi(\alpha\beta)$. Thus $P = y(\alpha\beta)^{-1}$ for some $y \in X\alpha\beta$. We then have $y \in X\beta$, so $Q\beta_* = y$ for some $Q \in \pi(\beta)$. Since $\beta \leq \gamma$, by Theorem 2.2 we have $Q\beta_* \in Q\gamma$. Consider

$$P(\alpha\beta)_* = y\beta^{-1}\alpha^{-1}\alpha\beta_* = Q\alpha^{-1}\alpha\beta_* = Q\beta_* \in Q\gamma = Q\alpha^{-1}\alpha\gamma = y\beta^{-1}\alpha^{-1}\alpha\gamma = P\alpha\gamma.$$

It follows by Theorem 2.2 that $\alpha\beta \leq \alpha\gamma$. Therefore $\alpha$ is left compatible with $\leq$ on $T_{SE}(X)$. ■

**Theorem 2.5.** Let $\alpha \in T_{SE}(X)$. Then $\alpha$ is right compatible with $\leq$ on $T_{SE}(X)$ if and only if for every $A \in X/E, A \in \pi(\alpha)$ or $|P| = 1$ for all $P \in \pi_A(\alpha)$.

***Proof.*** Assume that there exists $A \in X/E$ such that $A \notin \pi(\alpha)$ and $|P'| > 1$ for some $P' \in \pi_A(\alpha)$. By Proposition 1.2(1), we have $P' \subseteq A$. Since $A \notin \pi(\alpha)$, it follows that $P' \neq A$. We choose and fix elements $p' \in P'$ and $a \in A \setminus P'$. Then $p'\alpha = P'\alpha_*$ and $a\alpha \neq P'\alpha_*$. Now, define $\beta: X \to X$ by

$$x\beta = \begin{cases} a & \text{if } x = p'; \\ x & \text{otherwise.} \end{cases}$$

Let $x \in X$. If $x \neq p'$, then $(x, x\beta) = (x,x) \in E$. If $x = p'$, then $(x, x\beta) = (p', a) \in E$ since $p', a \in A$. Thus $\beta \in T_{SE}(X)$.

It easy to see that $\pi(I_X) = \{\{x\} : x \in X\}$ is a refinement of $\pi(\beta)$ where $I_X$ is the identity map on $X$. Moreover, $P\beta_* \in PI_X$ for all $P \in \pi(\beta)$. By Theorem 2.2, $\beta \leq I_X$. Since $(P'\alpha_*)(I_X\alpha)^{-1} = (P'\alpha_*)\alpha^{-1}I_X^{-1} = P'I_X^{-1} = P'$, we deduce $P' \in \pi(I_X\alpha)$. By the definition of $\beta$ and $P'\backslash\{p'\} \neq \emptyset$, we have that $P'\beta\alpha = (\{a\} \cup P'\backslash\{p'\})\alpha = \{a\alpha, P'\alpha_*\}$. Claim that $P' \nsubseteq Q$ for all $Q \in \pi(\beta\alpha)$. Suppose not, there exists $Q \in \pi(\beta\alpha)$ such that $P' \subseteq Q$. Since $\{a\alpha, P'\alpha_*\} = P'\beta\alpha$, it follows that $\{a\alpha, P'\alpha_*\} \subseteq Q\beta\alpha = \{Q(\beta\alpha)_*\}$ which is a contradiction. So we have the claim. This proves that $\pi(I_X\alpha)$ does not refine $\pi(\beta\alpha)$. By Theorem 2.2, we conclude that $\beta\alpha \nleq I_X\alpha$. Therefore $\alpha$ is not right compatible.

Conversely, suppose that for all $A \in X/E$, $A \in \pi(\alpha)$ or $|P| = 1$ for all $P \in \pi_A(\alpha)$. Let $\beta, \gamma \in T_{SE}(X)$ be such that $\beta \leq \gamma$. To show that $\beta\alpha \leq \gamma\alpha$ via Corollary 2.3, let $A \in X/E$. We consider two cases as follow.

**Case 1.** $A \in \pi(\alpha)$. Then $A\alpha_* = y$ for some $y \in X\alpha$. By Proposition 1.2(2), $A\beta \subseteq A$. Since $A\beta\alpha \subseteq A\alpha = \{y\}, A \subseteq y(\beta\alpha)^{-1} \in \pi(\beta\alpha)$. By Proposition 1.1, there exists $B \in X/E$ such that $y(\beta\alpha)^{-1} \subseteq B$. Then $A = B$ since $X/E$ is a partition of $X$. Hence $A = y(\beta\alpha)^{-1}$ which implies that $\pi_A(\beta\alpha) = \{A\}$. Similarly, we have that $\pi_A(\gamma\alpha) = \{A\}$. Hence $\pi_A(\gamma\alpha)$ refines $\pi_A(\beta\alpha)$. Moreover, let $P \in \pi_A(\beta\alpha) = \{A\}$. Then $P(\beta\alpha)_* = A(\beta\alpha)_* = y \in \{y\} = A\gamma\alpha = P\gamma\alpha$.

**Case 2.** $|P| = 1$ for all $P \in \pi_A(\alpha)$. Let $P \in \pi_A(\gamma\alpha)$. $P(\gamma\alpha)_* = y$ for some $y \in X\gamma\alpha$. Then $P\gamma \subseteq y\alpha^{-1}$. Since $y\alpha^{-1} \in \pi_A(\alpha)$, by assumption $|y\alpha^{-1}| = 1$. Let $y\alpha^{-1} = \{x\}$ for some $x \in X$. We then have $P\gamma = \{x\}$ and $P \cap A \neq \emptyset$, hence $P = x\gamma^{-1} \in \pi_A(\gamma)$. Since $\beta \leq \gamma$, by Corollary 2.3, $\pi_A(\gamma)$ refines $\pi_A(\beta)$. Hence $P \subseteq Q$ for some $Q \in \pi_A(\beta)$. This means that $P\beta \subseteq Q\beta = \{Q\beta_*\}$.

Now, we consider $P\beta\alpha \subseteq Q\beta\alpha = \{Q\beta_*\alpha\}$, thus $P \subseteq (Q\beta_*\alpha)(\beta\alpha)^{-1}$. Since $\emptyset \neq A \cap P \subseteq A \cap (Q\beta_*\alpha)(\beta\alpha)^{-1}, (Q\beta_*\alpha)(\beta\alpha)^{-1} \in \pi_A(\beta\alpha)$. This proves that $\pi_A(\gamma\alpha)$ refines $\pi_A(\beta\alpha)$.

Next, let $P \in \pi_A(\beta\alpha)$. Then $P(\beta\alpha)_* = y$ for some $y \in X$ which implies that $P\beta \subseteq y\alpha^{-1}$. By assumption, $y\alpha^{-1} = \{x\}$ for some $x \in X$, hence $P\beta = \{x\}$. Therefore $P = x\beta^{-1} \in \pi_A(\beta)$. It follows from $\beta \leq \gamma$ and Corollary 2.3, we have $P\beta_* \in P\gamma$. Hence $P(\beta\alpha)_* \in P\beta\alpha \subseteq (P\gamma)\alpha$.

From each case, we conclude that $\beta\alpha \leq \gamma\alpha$ by Corollary 2.3. This shows that $\alpha$ is right compatible with $\leq$ on $T_{SE}(X)$. ■

## REFERENCES


Kowol G, Mitsch H (1986). Naturally ordered transformation semigroups. Monatsh. Math., 102:115-138.

Ma M, You T, Luo Sh, Yang Y, Wang L (2010). Regularity and Green's relations for finite E-order-preserving transformations semigroups. Semigroup Forum 80:164-173.

Mitsch H (1986). A natural partial order for semigroups. Proc. Am. Soc. 97:384-388.

Sun L, Pei H, Cheng Z (2008). Naturally ordered transformation semigroups preserving an equivalence. Bull. Austral. Math. Soc. 78:117-128.