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Quantum fluctuations of the spacetime geometry in the spectral scheme[#]

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The spectral scheme for spacetime geometry is a totally new framework for quantitatively describing the spacetime geometry in terms of the spectra of a certain elliptic operator (typically the Laplacian operator) on the space in question. The central idea of the framework can be symbolically stated as "Let us *hear* the shape of the Universe!" There are several advantages of this framework compared to the traditional geometrical description in terms of the Riemannian metric. After sketching the basics of the spectral scheme, we give a new formula for the Einstein-Hilbert action, which is a central quantity for the general relativity theory, in terms of the spectral scheme. We then pay attention to its application to the quantum universes and see how the quantum fluctuations of spacetimes can be effectively described in terms of the spectral scheme.

Key words: General relativity, spacetime structures, spectral scheme, cosmology.

INTRODUCTION

The year 2016 is a special year for celebrating the centenary of the general theory of relativity, which has been presented by Albert Einstein in 1916 (Einstein, 1916). It might be, thus, an appropriate occasion to discuss some of modern attempts for pursuing the original dreams of Einstein.

We here focus on the spectral scheme for spacetime physics as one of such promising modern attempts along the line of Einstein. The spectral scheme for spacetime

geometry is a totally new framework, constructed by the author (Seriu, 1996, 2000a, b), which quantitatively describes the spatial geometry of the universe in terms of the spectra of elliptic operators (typically the Laplacian operator) on the space in question. Here the spectra of an elliptic operator defined on a space roughly correspond to the "sound properties of a drum" if we dare to search for some analogous concepts in daily life. Indeed there is a famous mathematical problem posed by a mathematician M. Kac, which states "Can one hear the shape of a drum?" (Kac, 1966). The mathematical question suggested by this phrase is whether and to what

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extent the geometrical information of a space ("the shape of a drum") can be inferred purely from the spectra of the space ("the sound of the drum"). Borrowing the phrase, the central idea of the spectral scheme can be symbolically stated as "Let us *hear* the shape of the Universe!"

THE BASICS OF THE SPECTRAL SCHEME

The spectral scheme is a rigorous theoretical scheme for analyzing the geometrical structures of the spacetime in terms of the spectra, that is, the "sound" of the universe, so to say, as mentioned previously.

There are several advantages of this framework compared to the standard description based on the Riemannian metric tensor.

One among them is that the spectra carry important geometrical information, both the local one and the global one in a unified manner, so that they are quite suitable for describing the *scale-dependent topology* of the spacetime (Visser, 1990; Seriu, 1993). Here the scale-dependent topology is a topological structure of the spacetime which becomes more and more topologically complicated as the observational scale becomes more and more microscopic.

It is generally believed that the quantum universes should possess this type of scale-dependent topological structures due to quantum fluctuations in the spacetime, which is usually called the "spacetime foam" picture (Wheeler, 1957). On the other hand, the standard framework heavily relies on the Riemannian metric tensor, which is basically a local quantity and is not convenient to describe the global topology of the space.

For another advantage of the spectral scheme (though deeply related to the above one), we recall a standard speculation that the microscopic spacetime structures get "fuzzy" due to quantum fluctuations near the Planck scale (1957). It is very difficult to describe such fluctuated geometries (including topological fluctuations) in terms of the standard metric description. We argued subsequently in the study that such quantum fluctuations in spacetime structures are naturally depicted in the spectral scheme. We also note that the spectra are diffeomorphism invariant quantities so that they carry purely geometrical information (that is, independent of the choice of the coordinate system) of the space. This property totally matches the spirit of relativity. On the contrary, the Riemannian metric is a tensor and it transforms according to the choice of the coordinate system, meaning that the metric carries not only purely geometrical information but also unphysical information on the choice of the coordinate system. Needless to say, the latter causes several complications for investigating geometry of the space, including quantum spacetimes. As is shown later, this point becomes important when we discuss the expression for the wave function of the Universe.

Now let $\mathcal{G} := (\Sigma, h_{ab})$ be the geometry of the $(D-1)$ -

dimensional¹ Riemannian manifold Σ with the metric h_{ab} . For concreteness, we assume that Σ is a compact manifold without boundaries and that h_{ab} is positive definite. We may consider \mathcal{G} as a mathematical model for the "space", that is the spatial section of the universe. Let Δ be the Laplacian operator on \mathcal{G} , defined by $\Delta f := h^{ab} \nabla_a \nabla_b f = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} h^{ab} \partial_b f)$ for a scalar smooth function f . Here ∇_a is the covariant derivative compatible with h_{ab} ; h^{ab} is the inverse matrix of h_{ab} ; $\sqrt{g} := \sqrt{\det h_{ab}}$. (We can also consider other elliptic operators. Here we confine ourselves to the Laplacian operator for concreteness). We then set the eigenvalue problem, $\Delta f_n = -\lambda_n f_n$, getting the solutions, (λ_n, f_n) ($n = 0, 1, 2, \dots$), where λ_n is the n -th

eigenvalue (or *spectrum*) and f_n is the n -th eigenfunction. (We number the spectra in an increasing order, $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$.)

Here the *spectra*, or the set of all (countably infinite) eigenvalues, $\{\lambda_n\}_{n=0,1,2,\dots}$, carry information on the geometry \mathcal{G} .

Let us thus use the spectra $\{\lambda_n\}_{n=0,1,2,\dots}$ of the Laplacian (or any other elliptic operator) as the new variables representing the geometry \mathcal{G} . This is the core idea of the spectral scheme. Let us call this representation of the geometry \mathcal{G} in terms of the spectra the *spectral representation of geometry*.

As mentioned previously, it might be helpful to imagine the geometry \mathcal{G} as though a musical drum. Then the eigenvalue problem $\Delta f_n = -\lambda_n f_n$ corresponds to the analysis of the vibration modes of the drum, where the n -th spectrum λ_n and the n -th eigenfunction f_n are analogous to, respectively, the n -th overtone (harmonics) and its vibration mode of the drum. Now the phrase "Can one hear the shape of a drum?" due to Kac (1966) is a question as to what extent one can infer the shape of a drum by just hearing its sound without looking at it. In mathematical terms, the question is rephrased as to what extent the geometrical information \mathcal{G} is captured by the spectra $\{\lambda_n\}_{n=0,1,2,\dots}$ of the Laplacian.

At this stage, it is appropriate to mention that there exist the so-called *isospectral manifolds*, that is, mutually diffeomorphism inequivalent manifolds with exactly the same spectra of the Laplacian (Kac, 1966; Chavel, 1984).

From the viewpoint of spacetime physics, however, there is no surprise in such manifolds and they do not cause any problem to the spectral scheme, as can be seen subsequently.

The physical interpretation of the isospectral manifolds is as follows. Symbolically speaking, each elliptic

¹Throughout this paper, the spacetime dimension is assumed to be D so that the space dimension is set

operator corresponds to, in a sense, one particular way of "hitting the drum" so that it detects some portion of geometrical information of the space ("the drum"). There is no surprise, then, for the existence of the isospectral manifolds because one elliptic operator such as the Laplacian extracts only some portion of the whole geometrical information. If we take into account all the elliptic operators allowed on the space, then, all the physically sensible geometrical information on the space should be extracted. If two spaces would have exactly the same spectral profile for all the elliptic operators, then, one would safely say that they are physically indistinguishable and effectively the same space (Seriu, 2000a). With this interpretation in mind, we can safely confine ourselves to the Laplacian operator as a typical elliptic operator throughout this paper.

Some more comments are in order here. As mentioned previously, we understand the spectra, $\{\lambda_n\}_{n=0,1,2,\dots}$, are numbered in an increasing order, $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$. Now the smallest eigenvalue λ_0 is always zero with its eigenfunction being $f \equiv const$. (The mode ($\lambda_0 = 0, f_0 \equiv const$) is usually called the "zero mode"). When the space in question is a compact one,

which is always assumed here, λ_n then corresponds to the smallest non-zero spectrum. Some spectra can be of the same value even though their eigenfunctions are mutually independent (This phenomenon is customarily called "degeneracy").

As it might be clear now, the spectral scheme is a rigorous theoretical scheme for analyzing the geometrical structures of the spacetime in terms of the spectra of the space, or the "sound" of the universe, so to say. Based on the spectral representation, we further need to invent theoretical tools for completing the scheme. Among them, the following three components are the most important ones.

- (1) The spectral distance between two spaces (Seriu, 1996)
- (2) The space of spaces (Seriu, 2000a)
- (3) The spectral evolution equations (Seriu, 2000b)

Here let us briefly sketch them one by one. The reader is referred to the original articles for more details (Seriu, 1996, 2000a, b).

The spectral distance between two spaces

We often need to compare two geometries \mathcal{G}_1 and \mathcal{G}_2 , judging whether and to what extent they are "similar" to each other.

For instance, in cosmology we try to investigate the dynamics of our Universe by choosing some model universe. The latter is a mathematical model for our

Universe and is usually much simpler with higher symmetries than the real Universe. Here the problem arises. Due to the highly nonlinear nature of the Einstein equation, there is no guarantee for the model universe to remain as a "good" model for the real Universe throughout the history of the Universe. If the Einstein equation possesses chaotic properties this could be the case and we would be forced to give up any reasonable predictions for the future or the past of our Universe. This fundamental, serious problem in cosmology is often called the "averaging problem"² and still remains unresolved. One of the main obstacles for resolving this fundamental problem resides in that there has been no standard way of quantitatively describing how the model universe is "close" or "similar" to the real Universe.

We thus need to define a suitable "distance" between two spaces. From the viewpoint of the spectral representation, however, it turns out that we can indeed introduce a sort of a distance between two spaces by comparing their spectra to each other. The basic idea might be symbolically stated that we compare the "shapes" of two "drums" by comparing their "sound" qualities.

Let $\mathcal{G} := (\Sigma, h_{ab})$ and $\mathcal{G}' := (\Sigma', h'_{ab})$ be two geometries, that is, two Riemannian manifolds³. Suppose $\{\lambda_n\}_{n=0,1,2,\dots}$ and $\{\lambda'_n\}_{n=0,1,2,\dots}$ are the spectra for \mathcal{G} and \mathcal{G}' , respectively, arranged in an increasing order ($0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$);

Then we define:

$$d_N(\mathcal{G}, \mathcal{G}') := \frac{1}{2} \sum_{n=1}^N \ln \left\{ \frac{1}{2} \left(\sqrt{\frac{\lambda_n}{\lambda'_n}} + \sqrt{\frac{\lambda'_n}{\lambda_n}} \right) \right\}, \quad (1)$$

which we call the *spectral distance* between \mathcal{G} and \mathcal{G}' . Here the zero-modes ($\lambda_0 = \lambda'_0 = 0$) are excluded from the summation. The summation is up to N to make $d_N(\mathcal{G}, \mathcal{G}')$ always finite.

Roughly speaking, the n -th spectrum λ_n reflects the geometrical information on the space \mathcal{G} of scale $\lambda_n^{-1/2}$, so that the smaller and smaller scale information is carried by the spectra with larger and larger indices n 's.

²Here is the sense of the term "the averaging problem". The geometry of the real Universe should be extremely complicated due to galaxies, stars, hydrogen clouds, etc. We expect that a model of the real Universe is obtained by suitably "averaging" the real complicated geometry at each point of the Universe and forming a smooth geometry with higher symmetries. However there is neither satisfactory way of taking the geometrical average of the Universe in a relativistically invariant manner nor any guarantee that the averaging procedure faithfully preserves the dynamical evolution of the real Universe due to highly nonlinear nature of the Einstein equation.

³It is important to note that even the dimensions and/or global topologies of \mathcal{G} and \mathcal{G}' can be different from each other. In general, however, the spectral distance between spaces with different dimensionality and/or different topologies becomes very large, which can be seen by Weyl's asymptotic formula (Chavel, 1984). There are still interesting cases where the spectral distance is surprisingly short even though the orientabilities of the spaces are opposite to each other (Seriu, 1996). We shall see one example of such cases at the end of this subsection.

Thus the introduction of the cutoff N corresponds to the coarse-grained comparison of the spaces \mathcal{G} and \mathcal{G}' , neglecting their smaller differences of the scale less than $\lambda_n^{-1/2}$. One can then treat N as a running parameter of $d_N(\mathcal{G}, \mathcal{G}')$, investigating the scale-dependent comparison between two spaces. Finally one can take $N \rightarrow \infty$ when the finest comparison is needed.

This type of scale-dependent comparison of geometries is very desirable considering, for instance, the scale-dependent topology (Seriu, 1993) based on the "spacetime foam" picture (Wheeler, 1957) as mentioned in the beginning part of this section.

It turns out that the spectral distance $d_N(\mathcal{G}, \mathcal{G}')$ has nice properties (Seriu, 1996). First it satisfies all the *axioms of distance* except for the axiom of triangle inequality. Furthermore the breakdown of the triangle inequality is a very mild one so that one can construct a nice space (called a *metrizable space*) of all spaces based on $d_N(\mathcal{G}, \mathcal{G}')$. Second, in the spectral distance, the differences in the lower modes (smaller index n , corresponding to the larger scale structures) are counted with more importance than the higher modes (larger index n , corresponding to the smaller scale structures). This is indeed a nice property since two shapes are usually judged as close to each other when their larger scale properties are similar rather than the smaller scale properties. This scale-sensitive property of $d_N(\mathcal{G}, \mathcal{G}')$ is desirable for describing the scale-dependent topology of the universe, which might be one of the important properties of quantum universes.

It is also quite important to note that the spectral distance $d_N(\mathcal{G}, \mathcal{G}')$ is not just a mathematical, abstract object, but is an actually computable quantity. For instance, the spectral distances $d_N(T^2_{(1)}, T^2_{(2)})$, $d_N(Klein_{(1)}, Klein_{(2)})$, $d_N(T^2, Klein)$ and $d_N(S^2, \mathbb{R}P^2)$ are rigorously computed including their $N \rightarrow \infty$ behaviors (Seriu, 1996). Here T^2 and S^2 stand for a 2-dimensional torus and a sphere, respectively, that are orientable surfaces, while $\mathbb{R}P^2$ is a 2-dimensional real projective space from which Klein's bottle ($\mathbb{R}P^2 \# \mathbb{R}P^2$) is constructed. The latter two surfaces are typical non-orientable surfaces. Here are some concrete results among numerous ones (For definiteness, the area of every surface here is normalized to unity). One example of the spectral distances between a standard torus and a thin torus is $d_N(T^2_{(1)}, T^2_{(2)}) \rightarrow 3.488$ (as $N \rightarrow \infty$), while the same between the standard torus and an extremely thin torus is $d_N(T^2_{(1)}, T^2_{(2)}) \rightarrow 68.02$ (as $N \rightarrow \infty$). One example of the spectral distances between a standard torus and Klein's bottle is $d_N(T^2, Klein) \rightarrow 0.4337$ (as $N \rightarrow \infty$), which is surprisingly short, considering that their orientabilities are opposite to each other.

The space of spaces

Since we now have a nice concept of distance between two spaces defined by the spectral profiles, we can construct an abstract space of "all spaces".

Let \mathcal{S}_N be the set of all possible Riemannian geometries⁴ $\mathcal{G} = (\Sigma, h_{ab})$ equipped with the spectral distance $d_N(\mathcal{G}, \mathcal{G}')$, where N is some suitably large integer. Then one can rigorously prove that \mathcal{S}_N forms a *metrizable space*, which means that \mathcal{S}_N can be regarded as a metric space by a suitable mild modification of $d_N(\mathcal{G}, \mathcal{G}')$ (Seriu, 2000a). Therefore \mathcal{S}_N forms a desirable space since it means that we can rigorously discuss, for instance, how "close" two spaces are and how a geometry dynamically develops in time within \mathcal{S}_N . We call \mathcal{S}_N the *space of spaces*.

As an example, suppose we want to investigate the dynamical evolution of a space \mathcal{G} (that is, the history of the universe \mathcal{G}). It is represented by a trajectory in \mathcal{S}_N starting from \mathcal{G} . To investigate whether a space \mathcal{G}' serves as a good model for the universe \mathcal{G} , we can in principle compare two trajectories, one for \mathcal{G} and the other for \mathcal{G}' . If two trajectories are close enough in \mathcal{S}_N , we can judge that the model \mathcal{G}' and its history are good representation of the universe \mathcal{G} .

In this manner, \mathcal{S}_N serves as a fundamental arena for analyzing cosmological questions quantitatively. It is for the first time in the history of cosmology such a "space of all universes" has been rigorously constructed.

The spectral evolution equations

Since the space (or the universe) evolves in time according to the Einstein equation, the spectra also evolve in time accordingly. This means that we can write down the Einstein equation in the spectral representation. We call such a spectral version of the Einstein equation the *spectral evolution equations* (Seriu, 2000b). The detailed treatment of the spectral evolution equations is given in the original paper (Seriu, 2000b).

Here is one of the important results obtained by the spectral evolution equations. The Hubble constant H is the most important cosmological parameter, which determines the age of one Universe. We should note that, however, the Hubble constant H is only definable in reference to a cosmological model, typically the Friedmann-Robertson-Walker model (the FRW model, for short). When we try to determine the value of H by observing the real Universe, thus, its estimated value is heavily influenced by the geometrical discrepancy between the real Universe and the FRW model and by the observational scale. Here the observational scale is typically determined by the wavelengths of the light

⁴For concreteness, they are assumed to be compact without boundaries with a positive definite metric. However there are no restrictions on the dimensionality or orientability.

signals detected in the observations.

We recall that the FRW model is an idealistic model of the Universe with the perfect spatial homogeneity and isotropy. Thus the inhomogeneity and anisotropy of the real Universe, even if they could be quite tiny, result in the geometrical discrepancy between the real Universe and the FRW model.

Now by means of the spectral evolution equations, we can derive the following formula (Seriu, 2000b);

$$H_n = H + \iota_n + \alpha_n \tag{2}$$

Here H_n is the effective Hubble constant at the observational scale $\lambda_n^{-1/2}$; ι_n and α_n are the inhomogeneity and anisotropy of the real Universe at the

same observational scale. Detailed expressions for ι_n

and α_n is given in Seriu (2000b). Since we estimate the age of the Universe by the Hubble constant, we need to understand the scale-dependence of the effective Hubble constant in more detail. Due to the detections of the gravitational waves, more detailed information on the inhomogeneity and anisotropy of our Universe is expected to be obtained in the near future. Thus the analyses based on Equation 2 shall be more and more important from now on. Just by this example, one would clearly see how the spectral scheme can be effectively applied to cosmology. Another, extremely important application of the spectral scheme to cosmology in the context of the notorious "averaging problem" is given subsequently.

The Einstein-Hilbert action in the spectral scheme

It is known that all the fundamental equations in nature found so far, including the Einstein equation, are derived from the *action-principle* by choosing a suitable *action* $S[q]$, that is a functional of the fundamental variables q 's.

It is not an exaggeration to say that the action $S[q]$ is the most important quantity in theoretical physics. Indeed, once the action $S[q]$ for a certain physical system is properly given, all the necessary physical information for the system can be derived from $S[q]$. This statement is true not only for the classical theories, but also for quantum theories, provided that the vacuum state is suitably fixed.

For the case of the Einstein's general theory of relativity (Einstein, 1916), the Einstein equation (for an empty spacetime) is derived from the action, known as the Einstein-Hilbert action,

$$S[g_{ab}] := \alpha \int_{\mathcal{M}} R \sqrt{} \tag{3}$$

Where \mathcal{M} is a D -dimensional spacetime manifold (that is,

a D -dimensional pseudo-Riemannian manifold) in question, R is the scalar curvature computed from the metric tensor g_{ab} , and α is a suitable constant. It means that a particular metric $g^{(0)}_{ab}$ giving the extremum of $S[g_{ab}]$ with respect to the variation of g_{ab} , characterized by $\delta S[g^{(0)}_{ab}] = 0$, is the solution for the Einstein equation

$$R_{ab} - \frac{1}{2} R g_{ab} = 0 \tag{4}$$

where R_{ab} is the Ricci curvature tensor (Wald, 1984).

When the cosmological constant Λ is taken into account, the action in Equation 3 is modified to

$$S[g_{ab}, \Lambda] := \alpha \int_{\mathcal{M}} (R - 2\Lambda) \sqrt{} \tag{5}$$

which yields instead of Equation 4,

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0 \tag{6}$$

The explicit expression for the constant α in Equations 3 or 5 is

$$\alpha := \frac{c^3}{4 S_{D-2} G}$$

where S_{D-2} is the $(D-2)$ -dimensional volume of the unit $(D-2)$ -sphere (regarded as dimension-free). For a standard case of $D=4$, S_{D-2} reduces to $s_2 = 4\pi$, so that $\alpha = \frac{c^3}{16\pi G}$. Let L , T and M represent, respectively, the physical dimensions of length, time and mass. Then, noting $[G] = [\frac{L^{D-1}}{MT^2}]$, we see that $[\alpha] = [\hbar] [L^{-(D-2)}]$ where $[\hbar]$ is the physical dimension of the Planck constant, i.e. the one for the action. Noting also that $[R] = [\Lambda] = [L^{-2}]$ and that the spacetime integral yields the physical dimension $[L^D]$, thus, the right-hand side of Equations 3 or 5 has the physical dimension of the action, as it should be.

In the spectral scheme, we consider the spectra $\{\lambda_n\}_{n=0,1,2,\dots}$ for the geometry $\mathcal{G} = (\mathcal{M}, g_{ab})$ are the

fundamental variables (rather than the metric tensor g_{ab} in the standard framework). Therefore it is desirable if we can express the Einstein-Hilbert action in terms of the spectra.

For this purpose, the D -dimensional Euclidean spacetime (\mathcal{M}, g_{ab}) is considered where \mathcal{M} is a compact manifold without boundaries⁵ and g_{ab} is a positive-definite metric on \mathcal{M} . This Euclidean setting is not a serious restriction on the theory. Indeed we are mostly interested

⁵It is of great interest to consider a manifold with spatial $(D-1)$ -manifolds as boundaries. However we here consider only the cases without boundaries for simplicity.

in the quantum universes in this context, and the Euclidean spacetimes are widely considered when studying quantum universes.

Now due to the heat kernel theory for the Laplacian operator, the following asymptotic formula is known (Chavel, 1984):

For a small positive parameter s , it follows

$$\sum_{n=0}^{\infty} e^{-\lambda_n s} = \frac{1}{(4\pi s)^{\frac{D}{2}}} \left\{ \int_{\mathcal{M}} \sqrt{+} + \frac{s}{6} \int_{\mathcal{M}} R \sqrt{+} + O(s^2) \right\}, \quad (7)$$

where we note that the zero-mode is also included in the summation on the left-hand side of Equation 7.

Comparing Equations 7 with 3, we realize the following: Let

$$\mathcal{S}(s) := 6\alpha(4\pi s)^{\frac{D}{2}} \sum_{n=0}^{\infty} e^{-\lambda_n s} \quad (8)$$

Here $[s]=[L^2]$, so that $[\mathcal{S}(s)] = [\hbar][L^2]$. Then we obtain the desired formula for the action,

$$S[\{\lambda_n\}] := \frac{d}{ds} \mathcal{S}(0) \quad (9)$$

In the same manner, comparing Equations 7 with 5, we get

$$S[\{\lambda_n\}, \Lambda] := \left(\frac{d}{ds} - \frac{1}{3} \Lambda \right) \mathcal{S}(0) \quad (10)$$

We can also consider instead of Equation 8,

$$\mathcal{S}_N(s) := 6\alpha(4\pi s)^{\frac{D}{2}} \sum_{n=0}^N e^{-\lambda_n s} \quad (11)$$

where the cutoff N has been introduced in the summation. Then the expressions Equations 9 and 10 with $\mathcal{S}(0)$ replaced by $\mathcal{S}_N(0)$ yield the scale-dependent actions $S_N[\{\lambda_n\}]$ and $S_N[\{\lambda_n\}, \Lambda]$ which naturally provides us with the coarse-grained description of spacetimes, taking into account the larger scale geometrical properties up to the scale N .

Now let us apply the action $S[\{\lambda_n\}]$ to get some insights on the quantum universes. As is well known, the quantum transition amplitude from the initial state i to the final state f can be estimated by the Euclidean path-integral

$$(f|i) \sim \int [dq] \exp -\frac{1}{\hbar} S[q] \quad (12)$$

For the spacetime physics, the quantum regime has not been well-understood so far, including the interpretation of quantum transition amplitudes for spacetimes, so that the rigorous treatment has not been established.

However, it is quite probable that the spacetimes or the universes cannot escape from the quantum principle on the microscopic scales just like the normal physical systems and that the formulas for the quantum

spacetimes should have several similarities with those for the normal physical systems.

Let us recall that a typical transition amplitude for quantum cosmology, or the "wave function of the Universe" as is customarily called, is generally believed to be given by (Hartle and Hawking, 1983).

$$\Psi \sim \int [d g_{ab}] \exp -\frac{1}{\hbar} S[g_{ab}], \quad (13)$$

where the functional integral is taken over all g_{ab} compatible with Euclidean manifolds without boundaries and the action $S[g_{ab}]$ is given by Equation 3 or Equation

5. One of the main difficulties of this expression is that g_{ab} contains unphysical components due to the general covariance of the theory (that is the gauge freedom of the general relativity theory caused by the freedom of the choice of coordinate systems) so that the functional integral over g_{ab} is not an easy task nor a procedure, requiring a full machinery of the gauge-fixing, the ghost fields, and so on, known as the Fadeev-Popov technique. Even worse, it turns out that this technique does not work satisfactorily for the case of gravity contrary to the cases of the non-Abelian gauge theories (Ryder, 1996).

In the spectral scheme, on the other hand, the transition amplitude (the "wave function of the universe") might be given as

$$\Psi \sim \int [d\{\lambda_n\}] \exp -\frac{1}{\hbar} S[\{\lambda_n\}], \quad (14)$$

where the integral is taken over all the spectra $\{\lambda_n\}$ compatible with Euclidean manifolds without boundaries and the action $S[\{\lambda_n\}]$ is given by Equations 9 or 10. As discussed previously, the spectra are diffeomorphism invariant quantities (that is, gauge-invariant quantities in this context), so that no difficulty arises in taking the integral over all the spectra $\{\lambda_n\}$. Thus the expression Equation 14 is conceptually much more transparent than the standard expression Equation 13.

The expression Equation 14 implies that the quantum spacetime is a superposition of various geometries and topologies corresponding to the various spectra $\{\lambda_n\}$. Furthermore if we replace $S[\{\lambda_n\}]$ with the scale-dependent action $S_N[\{\lambda_n\}]$ (with a cutoff scale N) it might be also possible to analyze the scale-dependent properties of quantum universes such as the spacetime-foam structures of spacetimes.

However there are several things to be cleared. One important issue is how to effectively perform the path-integral in Equation 14.

SOME COMMENTS ON THE APPLICATION OF THE SPECTRAL SCHEME

There are several important applications of the spectral

scheme. Among them is its applications to the so-called "averaging-problem" in cosmology (Seriu, 2000, 2001). As briefly discussed, the evolution of the universe governed by the Einstein equation, which is highly nonlinear, can be a chaotic one though there has been no detailed investigations on this subject so far. If this would be the case, even the slightest modification in the initial conditions results in a totally different future or past, which makes any prediction for the future nor the past on the Universe impossible. Though this is a very serious primary problem of cosmology itself, causing a fundamental doubt as "whether cosmology is possible" (Seriu, 2000, 2001), there has been no decisive argument on this problem. One of the main obstacles for tackling this issue is that there has been no useful framework for comparing geometries quantitatively. We need to compare the real Universe with its model and to judge how they are close to each other quantitatively. It is clear that the spectral scheme provides a fundamental framework for this type of analyses.

In Seriu (2001), the averaging problem in cosmology has been explicitly and quantitatively analyzed by means of the spectral scheme for the first time.

First the FRW model with small perturbations of inhomogeneity and anisotropy has been prepared, regarded as a mathematical representation of the "real Universe". Then the spectral distance between this modified FRW model and the pure FRW model has been investigated. In particular, with the help of the spectral evolution equations, the time evolution of the spectral distance has been estimated within the linear perturbation regime. The typical time-evolution of the spectral distance has turned out to be

$$d_N(\mathcal{G}, \mathcal{G}') \propto t^{-\frac{2(1-3\nu)}{3(1+\nu)}}, \tag{15}$$

when relatively shorter scale geometrical behaviors as well as the global features of the Universe are taken into account. Here $\nu = p/\rho$, the ratio of the pressure p and the energy density ρ , is a dimension-free parameter describing the matter content profile in the Universe ($0 \leq \nu \leq 1/3$ for normal matter); the horizon scale at the time of concern is the standard for "short" and "long". Similarly, we also get

$$d_N(\mathcal{G}, \mathcal{G}') \propto t^{-\frac{2(1-\nu)}{(1+\nu)}}, \tag{16}$$

when only the most global features of the Universe are taken into account.

Here the results Equations 15 and 16 show that at least within the linear regime, the spectral distance tends to converge in time, indicating that the FRW model remains as a good model at least within the time-scale in which the linear approximation is valid. In other words, cosmology as our attempt to understand the real Universe in reference to a model is guaranteed to be

valid at least within the linear regime. Needless to say, this is just the beginning of study and we should tackle the more important nonlinear phases.

It should be emphasized, however, that this analysis is the first rigorous quantitative attempt to solving the averaging problem in cosmology. There have been only preliminary studies along this line (Seriu, 2000, 2001), so that it is desirable that more intensive investigations are done on this fundamental problem making full use of the spectral scheme.

Regarding the possible chaotic property of the evolution of the universe, the author has been proposing a conjecture that *the chaotic properties of a system should get suppressed and tends to disappear as the number of degrees of freedom of the system approaches infinity*. Indeed virtually all the chaotic dynamical systems known so far have only the finite number of degrees of freedom. It is quite probable that the fundamental fact that the Universe is in principle an infinite existence is the key for understanding this problem. It might be helpful to mention that the above conjecture results from, in a sense, a strong belief that *we should be intelligent enough to reveal the mysteries of the Nature fully*, or in other words, that *the Nature should be kind enough to allow us to understand her fully*. If the evolution of the universe would be a chaotic one, we would be forced to give up guessing anything about the future and the past of our Universe; this could not be the case, considering the preliminary results (Equations 15 and 16) and the above-mentioned belief along with all the great triumphs of theoretical physics in history, including the triumph of the general theory of relativity by Einstein in 1916.

By these considerations, the above-mentioned conjecture on the relation between the chaotic property and the number of degrees of freedom seems the only reasonable possibility. The belief on the human intelligence combined with logical deductions turned out to have a power of yielding a meaningful conjecture and one would even imagine regarding the belief as one of the fundamental principles in physics. The author has coined this belief, regarded as a principle, the "Faust principle", named after the main character of Goethe's famous novel, who dreams of understanding the whole aspects of life.

As sketched briefly previously, the spectral scheme may also be an effective tool for studying quantum spacetimes. This is because the spectra naturally carry the information on the global topology of a spacetime as well as other geometrical information. In quantum spacetimes and quantum universes it is expected that spacetimes with various topologies are appearing and disappearing by the quantum fluctuations. The spectral scheme is expected to be effective for dealing with such situations. One such problem is the quantum decoherence between two universes with different topologies. There are some evidences that the spectral distance between two universes can be a useful measure

of their quantum decoherence (Seriu, 1996). This type of analysis in quantum universes has not yet been explored enough and more investigations are desirable.

Finally we should mention that, contrary to its drastically different appearance from a standard one, the spectral scheme is in principle widely applicable to problems in spacetime physics and cosmology in so far as they can be dealt with by the standard framework based on the metric tensor. This is because spacetimes which can be handled by the standard framework mostly possess several spatial symmetries and because the spectra of the spaces with high symmetries can be explicitly computed. Furthermore, as we have seen above, the spectral scheme can deal with the problems with which the standard framework cannot handle, such as the averaging problem in cosmology, so that it is clear that the spectral scheme has its own advantages over the standard framework.

In this paper we have seen only some of the results provided by the spectral scheme and we expect that much more intriguing results are awaited.

CONCLUSION

In this paper, we have reviewed the basics of the spectral scheme whose motto is "Let us *hear* the shape of the Universe!" Then we have presented some recent developments of the scheme including the spectral expression of the Einstein-Hilbert action and its application to quantum spacetimes.

Conflict of Interests

The author has not declared any conflict of interests.

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