Scientific Research and Essays Vol. 6(19), pp. 4021-4028, 8 September, 2011
Available online at http://www.academicjournals.org/SRE
DOI: 10.5897/SRE10.1060
ISSN 1992-2248 ©2011 Academic Journals

Full Length Research Paper

Planning step-stress life tests with type-ii censored data

Ali A. Ismail

King Saud University, College of Science, Department of Statistics and Operations Research, P. O. BOX 2455, Riyadh 11451, Saudi Arabia. E-mail: aismail100@yahoo.com. Tel: 0096614676269. Fax: 0096614673036.

Accepted 8 August, 2011

This study discusses the point and interval estimations of two-parameter Gompertz distribution under partially accelerated life tests with Type-II censoring. Also, the study considers the optimal design problem in the case of time step-stress model. The maximum likelihood estimators and approximate confidence intervals of the model parameters are derived. Also, optimum test plans for the time step-stress life tests are developed. For illustration, a simulation study is provided.

Key words: Gompertz distribution, partially accelerated life test, maximum likelihood estimation, Newton-Raphson method, fisher information matrix, Monte Carlo simulation, type-II censoring, optimal design, generalized asymptotic variance.

INTRODUCTION

With today's high technology, some life tests result in none or very few failures by the end of the test. In such cases, an approach is to do life test at higher-than-usual stress conditions in order to obtain failures quickly. Such an approach can be represented by accelerated life test (ALT) or partially accelerated life test (PALT). ALT is often used for reliability analysis. In ALT, test units are run at higher-than-usual stress conditions. It can be applied only when a model relating life length to stress is known. If such a model is unknown or cannot be assumed, another approach can be used which is PALT. In PALT, test units are run at both usual and higher-than-usual stress conditions. The stress loading in a PALT can be applied by various ways. They include step stress, constant stress and random stress. Nelson (1990) discussed their advantages and disadvantages. One way to accelerate failure is step-stress which increases the stress applied to test product in a specified discrete sequence. Generally, as indicated by Xiong and Ji (2004), a test unit starts at a specified low stress. If the unit does not fail at a specified time, stress on it is raised and held at a specified time. Stress is repeatedly increased and held, until the test unit fails or a censoring point is reached. In this paper, a simple time-step stress PALT that uses only two stress levels is considered. Under step-stress PALT, a test unit is first run at normal use condition and, if it does not fail for a specified time τ, then it is run at accelerated use condition until failure occurs or the observation is censored. The objective of such experiment is to collect more failure-times data in a limited time without necessarily using a high stress to all test units. In the literature, there are some studies on such partially accelerated life tests. Goel (1971) considered the estimation problem of the acceleration factor using both maximum likelihood and Bayesian methods for items having exponential distribution and uniform distribution. The estimates of the parameters of the lifetime distributions were obtained in the case of step-stress PALT under complete sampling. DeGroot and Goel (1979) used the Bayesian approach with different loss functions to estimate the parameters of the exponential distribution and the acceleration factor for step-stress PALT in the case of complete sampling. Also, PALT was studied with type-I censored data. For example, Bai and Chung (1992) used the maximum likelihood method to estimate the scale parameter and the acceleration factor for exponentially distributed lifetimes in the case of step-stress PALT. They also considered the problem of optimally designing PALT that terminates at a predetermined time. Bai et al. (1993) considered the estimation problem of parameters for items having lognormally distributed lives. The parameters of this life time distribution and the acceleration factor were estimated by the method of maximum likelihood in step PALT. Abdel-Ghaly et al. (1997) used the Bayesian approach for estimating the Weibull distribution parameters given that the shape parameter is known.

Madi (1997) applied the Gibbs sampling approach to the partially accelerated life testing (PALT). This sampling approach was proposed as a general method for Bayesian calculations. He derived empirical Bayes estimators for the failure of the exponential lifetime distribution under normal conditions. Abdel-Ghaly et al. (2002) studied both the estimation and optimal design problems for the Pareto distribution under step-stress PALT with type-I censoring. Abdel-Ghani (2004) considered the estimation problem of the log-logistic distribution parameters under step PALT in the case of type-I censored data. Recently, Aly and Ismail (2008) studied the optimal design problem of step-stress PALT in the case of Weibull distribution with type-I censored data. More recently, Ismail (2010) applied the Bayesian approach to the estimation problem in the case of step stress partially accelerated life tests with two stress levels and type-I censoring assuming the two-parameter Gompertz distribution as a lifetime model. This paper concentrates on both estimation and optimal design problems in the case of the two-parameter Gompertz distribution under step-stress PALT using type-II censored data.

THE MODEL AND TEST METHOD

Notations

n, total number of test items in a PALT; $Y_{(r)}$, the time of rth failure at which the experiment is terminated; T, lifetime of an item at normal use condition; Y, total lifetime of an item in a step PALT; $f(t)$, probability density function at time t at normal use condition; $R(t)$, reliability function at time t at normal use condition; $h(t)$, hazard (failure) rate at time t at normal use condition; β, acceleration factor ($\beta > 1$); τ, stress change-time in a step PALT ($\tau \le Y_{(r)}$); GAV, generalized asymptotic variance; MLEs, maximum likelihood estimates/estimators; MSE, mean square error; CIs, confidence intervals; IW, interval width; $\wedge$, implies a maximum likelihood estimate; $\downarrow$ (.), evaluated at (.); θ, α, the two parameters of Gompertz distribution ($\theta > 0$ and $\alpha > 0$); y_i, observed value of the total lifetime Y_i of item i , $i = 1, \ldots, n$; n_u, n_a, numbers of items failed at normal use and accelerated use conditions, respectively; n_u^*, n_a^*, optimal numbers of items failed at normal use and accelerated use conditions, respectively; $y_{(1)} \le \ldots \le y_{(n_u)} \le \tau \le y_{(n_u+1)} \le \ldots \le y_{(n_u+n_a)} \le y_{(r)}$ ordered failure times.

The gompertz distribution

The Gompertz distribution plays an important role in modeling survival times, human mortality and actuarial tables. As indicated by Walker and Adham (2001), the Gompertz distribution has many applications, particularly in medical and actuarial studies. According to the literature, the Gompertz distribution was formulated by Gompertz (1825) to fit mortality tables. Many authors have contributed to the statistical methodology and characterization of this distribution (Read 1983; Gordon, 1990; Makany, 1991; Rao and Damaraju, 1992; Franses, 1994; Wu and Lee, 1999). Garg et al. (1970) studied the properties of the Gompertz distribution and obtained the maximum likelihood estimates for the parameters. Osman (1987) derived a compound Gompertz model by assuming that one of the parameters of the Gompertz distribution is a random variable following the gamma distribution. He studied the properties of compound of Gompertz distribution and suggested its use for modeling lifetime data and analyzing the survivals in heterogeneous populations. Chen (1997) developed an exact confidence interval and an exact joint confidence region for the parameters of the Gompertz distribution under type-II censoring. According to Jaheen (2003), the Gompertz distribution has been used as a growth model, especially in epidemiological and biomedical studies. However, there has been little recent work on the Gompertz in comparison with its early investigation. The Gompertz distribution can be considered as a theoretical distribution of survival times. In this paper, the lifetimes of the test items are assumed to follow a Gompertz distribution with probability density function (pdf) as follows:

$$f(t) = \theta \exp\{\alpha t - (\theta/\alpha)[\exp(\alpha t) - 1]\}, \quad t > 0,\ \theta > 0,\ \alpha > 0 \qquad (1)$$

This distribution does not seem to have received enough attention, possibly because of its complicated form (Garg et al., 1970). It is worth noting that when $\alpha \to 0$, the Gompertz distribution will tend to an exponential distribution (Wu et al., 2003). The two-parameter Gompertz model is a commonly used survival time distribution in actuarial science and reliability and life testing (Ananda et al., 1996). There are several forms for the Gompertz distribution given in the literature. Some of these are given in Johnson et al. (1994). The pdf formula given in Equation 1 is the commonly used form and it is unimodal. It has positive skewness and an increasing hazard rate function. In addition, the Gompertz distribution can be interpreted as a truncated extreme value type-I distribution (Johnson et al., 1995). The reliability function of the Gompertz distribution takes the form:

$$R(t) = \exp\{-(\theta/\alpha)[\exp(\alpha t) - 1]\}, \qquad (2)$$

and the corresponding hazard rate is given by:

$$h(t) = \theta \exp(\alpha t), \qquad (3)$$

Thus, the hazard rate increases exponentially over time.

The test method

In the case of step-stress PALT, the test procedure and its assumptions are described as follows:

Test procedure

i) Each of the n test items is first run at normal use condition.
ii) If it does not fail at normal use condition by a pre-specified time τ, then it is put on accelerated use condition and run until it fails or the

test is terminated.

That is, if the item has not failed by some pre-specified time τ (τ is called stress change-time), the test condition is switched to a higher level of stress and it is continued until failure occurs or the observation is censored. The effect of this switch is to multiply the remaining lifetime of the item by the inverse of an acceleration factor β, which is the ratio of the hazard rate at accelerated condition to that at normal use condition ($\beta > 1$). Thus, the total lifetime of a test item denoted by Y passes through two stages, the first stage is the normal use condition and the second one is the accelerated use condition, respectively.

Assumptions

1) The total lifetime Y of an item is as follows:

$$Y = \begin{cases} T & \text{if } T \le \tau \\ \tau + \beta^{-1}(T - \tau) & \text{if } T > \tau, \end{cases} \tag{4}$$

Where T is the lifetime of an item at normal use condition. This model is called the tampered random variable (TRV) model. It was proposed by DeGroot and Goel (1979).

2) The lifetimes $Y_1, \ldots, Y_n$ of the n test items are independent and identically distributed random variables (*i.i.d. r.v.'s*).

MAXIMUM LIKELIHOOD ESTIMATION OF THE MODEL PARAMETERS

As indicated by Grimshaw (1993), the ML method is commonly used for most theoretical models and kinds of censored data. Although the exact sampling distribution of maximum likelihood estimators (MLEs) is sometimes unknown, MLEs have the desirable properties of being consistent and asymptotically normal for large samples. The lifetime of test unit is assumed to follow the two-parameter of Gompertz distribution with pdf given in Equation 1 (Garg et al., 1970). Therefore, the probability density function of total lifetime Y of an item in a step-stress PALT can be given by:

$$f(y) = \begin{cases} f_1(y) & \text{if } 0 < y \le \tau \\ f_2(y) & \text{if } y > \tau \end{cases} \tag{5}$$

Where,

$$f_1(y) = \theta \exp\{\alpha\, y - (\theta/\alpha)[\exp(\alpha\, y) - 1]\};$$

Which is an equivalent form to Equation 1,

$$f_2(y) = \beta\theta \exp\{\alpha[\beta(y-\tau)+\tau] - (\theta/\alpha)[\exp(\alpha[\beta(y-\tau)+\tau]) - 1]\}$$

$f_2(y)$ is obtained by the transformation-variable technique using Equations 1 and 4.

Point estimation

The observed values of the total lifetime Y are given by:

$y_{(1)} \le \ldots \le y_{(n_u)} \le \tau \le y_{(n_u+1)} \le \ldots \le y_{(r)}$; $r = n_u + n_a$

Since the total lifetimes $Y_1, \ldots, Y_n$ of n items are i.i.d.r.v.'s, then the general form of the total likelihood function for them can be written as:

$$L(.) \propto \prod_{i=1}^{n_u} f_1(y_i) \times \prod_{i=1}^{n_a} f_2(y_i) \times \prod_{i=1}^{n_c} R(y_{(r)}),$$

Where $n_c = n - r$.

Therefore, the likelihood function of the sample is given by:

$$\begin{aligned} L(\beta,\theta,\alpha) \propto & \prod_{i=1}^{n_u} \theta \exp\{\alpha y_i - (\theta/\alpha)[\exp(\alpha y_i) - 1]\} \\ & \times \prod_{i=1}^{n_a} \beta\theta \exp\{\alpha[\beta(y_i-\tau)+\tau] - (\theta/\alpha)[\exp(\alpha[\beta(y_i-\tau)+\tau]) - 1]\} \\ & \times \prod_{i=1}^{n_c} \exp\{-(\theta/\alpha)[\exp(\alpha[\beta(y_{(r)}-\tau)+\tau]) - 1]\} \end{aligned} \tag{6}$$

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. So, the natural logarithm of the likelihood function can be written as:

$$\begin{aligned} \ln L = & r \ln\theta + n_a \ln\beta + \alpha\{\sum_{i=1}^{n_u} y_i + \sum_{i=1}^{n_a} [\beta(y_i-\tau)+\tau]\} \\ & -(\theta/\alpha)\{\sum_{i=1}^{n_u} [\exp(\alpha y_i) - 1] + \sum_{i=1}^{n_a} [\exp(\alpha[\beta(y_i-\tau)+\tau]) - 1] \\ & + n_c[\exp(\alpha[\beta(y_{(r)}-\tau)+\tau]) - 1]\} \end{aligned} \tag{7}$$

The first derivatives of the natural logarithm of the total likelihood function in (7) with respect to β, θ and α are given by:

$$\frac{\partial \ln L}{\partial \beta} = \frac{n_a}{\beta} + \alpha \sum_{i=1}^{n_a} (y_i - \tau) - \theta \psi_1 \quad , \tag{8}$$

Where,

$$\psi_1 = \sum_{i=1}^{n_a} \{(y_i - \tau)\exp(\alpha[\beta(y_i - \tau) + \tau])\}$$
$$+ n_c(\eta - \tau)\exp(\alpha[\beta(y_{(r)} - \tau) + \tau])$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{r}{\theta} - \frac{\psi_2}{\alpha} \quad , \tag{9}$$

Where,

$$\psi_2 = \sum_{i=1}^{n_u} [\exp(\alpha\, y_i) - 1] + \sum_{i=1}^{n_a} [\exp(\alpha[\beta(y_i - \tau) + \tau]) - 1]$$
$$+ n_c[\exp(\alpha[\beta(y_{(r)} - \tau) + \tau]) - 1] \; .$$

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{n_u} y_i + \sum_{i=1}^{n_a} [\beta(y_i - \tau) + \tau] - (\theta / \alpha^2)[\alpha \psi_3 - \psi_2] \quad , \tag{10}$$

Where,

$$\psi_3 = \frac{\partial \psi_2}{\partial \alpha} = \sum_{i=1}^{n_u} [y_i \exp(\alpha y_i)] + \sum_{i=1}^{n_a} \{[\beta(y_i - \tau) + \tau]\exp(\alpha[\beta(y_i - \tau) + \tau])$$
$$+ n_c[\beta(y_{(r)} - \tau) + \tau][\exp(\alpha[\beta(y_{(r)} - \tau) + \tau])] \; .$$

From Equation 9, after equating it to 0, the maximum likelihood estimate of θ can be given by:

$$\hat{\theta} = \frac{r\hat{\alpha}}{\psi_2} \quad . \tag{11}$$

By substituting for θ into the two Equations 8 and 10 and equating each of them to zero, the system equations are then reduced to the following two non-linear equations:

$$\frac{n_a}{\hat{\beta}} + \hat{\alpha} \sum_{i=1}^{n_a} (y_i - \tau) - \frac{r\hat{\alpha}\psi_1}{\psi_2} = 0 \quad , \tag{12}$$

And

$$\sum_{i=1}^{n_u} y_i + \sum_{i=1}^{n_a} \hat{\beta}[(y_i - \tau) + \tau] - \frac{r(\hat{\alpha}\psi_3 - \psi_2)}{\hat{\alpha}\psi_2} = 0 \quad . \tag{13}$$

Obviously, it is very difficult to obtain a closed-form of solution for the two non-linear Equations 12 and 13. So, iterative procedures must be used to solve these equations, numerically. The Newton-Raphson method is used to obtain the MLEs of β and α. Thus, once the values of $\hat{\beta}$ and $\hat{\alpha}$ are determined, an estimate of θ is easily obtained from (11).

Interval estimation

The observed Fisher information matrix, as well as the asymptotic variance–covariance matrix of the MLEs is derived. Approximate confidence intervals (CIs) for the parameters based on normal approximation to the asymptotic distribution of MLEs are derived. As indicated by Vander Wiel and Meeker (1990), the most common method to set confidence bounds for the parameters is to use the large-sample (asymptotic) normal distribution of the ML estimators. To construct a confidence interval for a population parameter λ; assume that $L_\lambda = L_\lambda(y_1, \ldots, y_n)$ and $U_\lambda = U_\lambda(y_1, \ldots, y_n)$ are functions of the sample data $y_1, \ldots, y_n$ such that:

$P_\lambda(L_\lambda \le \lambda \le U_\lambda) = \gamma$, (14)

Where the interval $[L_\lambda, U_\lambda]$ is called a two-sided γ 100% confidence interval for λ. L_λ and U_λ are the lower and upper confidence limits for λ, respectively. The random limits L_λ and U_λ enclose λ with probability γ. Asymptotically, the maximum likelihood estimators are consistent and normally distributed. Therefore, the two-sided approximate γ 100% confidence limits for a population parameter λ can be constructed such that:

$$\mathrm{P}[-z \le \frac{\hat{\lambda} - \lambda}{\sigma(\hat{\lambda})} \le z] \cong \gamma, \tag{15}$$

Where z is the [100(1-γ/2)]th standard normal percentile and $\sigma(\hat{\lambda})$ is the standard deviation of the point estimate of the parameter λ. Thus, the two-sided approximate γ 100% confidence limits for β, θ and α are given respectively by:

$$L_\beta = \hat{\beta} - z\sigma(\hat{\beta}) \; U_\beta = \hat{\beta} + z\sigma(\hat{\beta})$$
$$L_\theta = \hat{\theta} - z\sigma(\hat{\theta}) \; U_\theta = \hat{\theta} + z\sigma(\hat{\theta}) \tag{16}$$

$$L_\alpha = \hat{\alpha} - z\sigma(\hat{\alpha}) \; U_\alpha = \hat{\alpha} + z\sigma(\hat{\alpha})$$

In relation to the asymptotic variance-covariance matrix of the MLE of the parameters, it can be approximated by numerically inverting the observed Fisher-information matrix. The observed Fisher-information matrix is composed of the negative second derivatives of the natural logarithm of the likelihood function evaluated at the MLEs. It can be given by the following matrix:

$$F = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \beta^2} & -\frac{\partial^2 \ln L}{\partial \beta \partial \theta} & -\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} \\ -\frac{\partial^2 \ln L}{\partial \theta \partial \beta} & -\frac{\partial^2 \ln L}{\partial \theta^2} & -\frac{\partial^2 \ln L}{\partial \theta \partial \alpha} \\ -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} & -\frac{\partial^2 \ln L}{\partial \alpha^2} \end{bmatrix} \downarrow (\hat{\beta}, \hat{\theta}, \hat{\alpha}) \qquad (17)$$

The elements of the matrix *F* in (17) can be expressed by the following equations:

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n_a}{\beta^2} - \theta\alpha\{\sum_{i=1}^{n_a}[(y_i-\tau)^2 \exp(\alpha[\beta(y_i-\tau)+\tau])] + n_c(y_{(r)}-\tau)^2 \exp(\alpha[\beta(y_{(r)}-\tau)+\tau])\} \;, \qquad (18)$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{r}{\theta^2} \;, \qquad (19)$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -(\theta/\alpha^2)[\alpha\psi_4 - \psi_3] + (\theta/\alpha^4)[\alpha^2\psi_3 - 2\alpha\psi_2] \;, \qquad (20)$$

Where,

$$\psi_4 = \frac{\partial \psi_3}{\partial \alpha} = \sum_{i=1}^{n_u}[y_i^2 \exp(\alpha\, y_i)] + \sum_{i=1}^{n_a}\{[\beta(y_i-\tau)+\tau]^2 \exp(\alpha[\beta(y_i-\tau)+\tau])\} + n_c[\beta(y_{(r)}-\tau)+\tau]^2[\exp(\alpha[\beta(y_{(r)}-\tau)+\tau])] \;.$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \theta} = -\psi_1 \;, \qquad (21)$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} = \sum_{i=1}^{n_a}(y_i-\tau) - \theta[\sum_{i=1}^{n_a}\{(y_i-\tau)[\beta(y_i-\tau)+\tau]\exp(\alpha[\beta(y_i-\tau)+\tau])\} + n_c(y_{(r)}-\tau)[\beta(y_{(r)}-\tau)+\tau]\exp(\alpha[\beta(y_{(r)}-\tau)+\tau])] \;, \qquad (22)$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial \alpha} = -(1/\alpha^2)[\alpha\psi_3 - \psi_2] \;. \qquad (23)$$

Therefore, the maximum likelihood estimators of β, θ and α have an asymptotic variance-covariance matrix obtained by inverting the Fisher information matrix defined in equation (17). The observed Fisher information matrix enables us to construct CIs for the parameters based on the limiting normal distribution through simulation.

OPTIMUM TEST PLAN

Now, for the optimal design stage of the test, a new experiment with test units different from those tested in the stage of parameter estimation is conducted. The current aim is to obtain the optimal value of τ based on the outputs of the stage of parameter estimation that are in the same time considered inputs to the optimal design stage of the test. It is worth noting that the stress change-time τ is a prespecified time for the stage of parameter estimation. But for the optimal design stage of the test, τ is considered a switching parameter to be optimally determined according to a certain optimality criterion. Here, the problem of optimally designing a simple time-step-stress PALT is considered which terminates after a pre-specified number of failures. Optimum test plans for products having a two-parameter Gompertz lifetime distribution are developed. The optimum criterion is to find the optimal stress change-time τ^* such that the generalized asymptotic variance (GAV) of the MLEs of the model parameters at normal use condition is minimized. The GAV of the MLEs of the model parameters is the reciprocal of the determinant of *F*. That is:

$$GAV\,(\hat{\beta}, \hat{\theta}, \hat{\alpha}) = \frac{1}{|F|} \qquad (24)$$

The minimization of the GAV over τ solves the following equation:

$$\frac{\partial\, GAV}{\partial\, \tau} = 0 \qquad (25)$$

In general, the solution to (16) is not in a closed form and therefore requires a numerical method such as Newton-Raphson. The Newton-Raphson method is applied to obtain the optimal stress-change time τ^* which minimizes the GAV.

SIMULATION STUDIES

The main objective of this simulation study is to make a

Table 1. The MLEs of the parameters, the associated MSE and the IW of the model parameters for different sized samples using type-II censoring in step-stress PALT with $\tau = 2.5$ and $r = 0.70\ n$.

n	(β, θ, α) Parameter	(2, 0.5, 0.6)			(5, 0.8, 1.4)		
		Estimate	**MSE**	**IW**	**Estimate**	**MSE**	**IW**
100	β	2.3473	0.2157	1.6331	5.4874	0.6104	0.9581
	θ	0.5406	0.0381	1.1273	0.9623	0.0537	0.7945
	α	0.6435	0.0243	0.8724	1.6725	0.2534	0.6429
200	β	2.2510	0.1684	1.4892	5.4323	0.4735	0.7615
	θ	0.5398	0.0267	1.0544	0.9106	0.0352	0.7211
	α	0.6275	0.0154	0.6091	1.6342	0.1916	0.5378
300	β	2.1938	0.1132	1.3675	5.3711	0.3286	0.6311
	θ	0.5257	0.0183	0.8976	0.8754	0.0267	0.5133
	α	0.6114	0.0117	0.4111	1.5482	0.1461	0.3921
400	β	2.1256	0.0651	1.2944	5.2458	0.2166	0.4392
	θ	0.5231	0.0118	0.8361	0.8474	0.0243	0.3206
	α	0.6089	0.0068	0.2846	1.4633	0.0645	0.2744
500	β	2.0722	0.0290	1.1167	5.1939	0.1644	0.2685
	θ	0.5164	0.0068	0.7922	0.8287	0.0177	0.1955
	α	0.6062	0.0044	0.1275	1.4301	0.0376	0.1271
800	β	2.0310	0.0211	1.0231	5.1153	0.0788	0.1746
	θ	0.5086	0.0041	0.6535	0.8085	0.0122	0.1153
	α	0.6032	0.0027	0.0761	1.4216	0.0214	0.0822
1000	β	2.0046	0.0173	0.9261	5.0352	0.0347	0.1172
	θ	0.5047	0.0028	0.4106	0.8003	0.0104	0.0781
	α	0.6011	0.0012	0.0354	1.4008	0.0210	0.0301

numerical investigation for illustrating the theoretical results of both estimation and optimal design problems given in this paper. Considering the type-II censoring, several data sets are generated from Gompertz distribution for different combinations of the true parameter values of β, θ and α. The true parameter values used here are (2, 0.5, 0.6) and (5, 0.8, 1.4). Different samples sizes (n = 100, 200, 300, 400, 500, 800 and 1000) are considered using 10000 replications for each sample size. The Newton-Raphson method is used for obtaining the MLEs of β, θ and α. Therefore, the derived nonlinear logarithmic likelihood equations in (12) and (13) are solved iteratively. Once the values of $\hat{\beta}$ and $\hat{\alpha}$ are determined, an estimate of the parameter θ is easily obtained from Equation 11. For different sample sizes and different true values of the parameters, the MLEs of the model parameters, their mean square error (MSE) and the interval width (IW) of the parameters are reported in Table 1. The results of simulation studies provide insight into the sampling behavior of the estimators. These results indicate that the ML estimates approximately the true values of the parameters as the sample size n increases. Also, as shown from the numerical results, the MSE of the estimators decrease as the sample size n is getting to be larger. Moreover, the IW of the model parameters is shown to be narrower as the sample size n increases. Table 2 presents the results of optimum test plans.

The numerical results reported in Table 2 demonstrate that the PALT model is appropriate because there have failed items in both stages. That is, testing not only at normal use condition but also at accelerated condition. Also, Table 2 presents the optimal GAV which is numerically obtained with τ^* in place of τ for different sized samples. As indicated from the results, the optimal

Table 2. The results of optimal design of step-stress PALT for different sized samples using type-II censoring with $r = 0.70\ n$.

(β, θ, α)	(2, 0.5, 0.6)				(5, 0.8, 1.4)			
n	τ^*	n^*_u	n^*_a	Optimal GAV	τ^*	n^*_u	n^*_a	Optimal GAV
100	1.7661	45	25	0.4922	1.2653	43	27	0.4511
200	1.8644	97	43	0.4151	1.2871	92	46	0.3873
300	1.8952	158	52	0.3477	1.3398	152	58	0.2465
400	1.9107	207	73	0.2588	1.3462	204	76	0.1782
500	1.9232	268	82	0.1162	1.5074	258	92	0.0931
800	1.9410	421	139	0.0471	1.5422	416	144	0.0217
1000	1.9462	543	157	0.0133	1.5715	527	173	0.0112

GAV decreases as the sample size increases.

Conclusion

This paper considered the problems of estimation and optimally designing simple time-step stress PALT for the Gompertz distribution under type-II censored data. The MLEs and IW of the model parameters were obtained. Also, optimum test plans were developed under the assumptions of Gompertz lifetimes of test units and type-II censoring. The minimization of the GAV of the MLEs of model parameters was adopted as an optimality criterion. It is concluded that the PALT model is a suitable scheme. It enables us to save time and money in a limited time without necessarily using a high stress to all test units. In practice, the optimum test plans are important for improving the level of precision in parameter estimation and thus improving the quality of the inference. So, statistically, optimum plans are needed, and the experimenters are advised to use it for estimating the life distribution at design stress. The usefulness of the optimal design lies in the fact that it can serve as a benchmark for comparison with other designs. As a future work, the problem of designing time-step stress PALT based on progressively censored data will be studied for this distribution. Also, the case of multi-stress life tests will be considered under the same distribution.

ACKNOWLEDGEMENTS

This project was supported by King Saud University, Deanship of Scientific Research, College of Science Research Center. The author would like to thank the editors and the referees for their helpful comments and corrections which led to improvements of an earlier version of this article.

REFERENCES

Abdel-Ghaly AA, Attia AF, Abdel-Ghani MM (1997). The Bayesian Estimation of Weibull Parameters in Step Partially Accelerated Life Tests with Censored Data. Proceedings of the 32st Annual conference of statistics, computer sciences and operation research, ISSR, Cairo University, pp. 45-59.

Abdel-Ghaly AA, El-Khodary EH, Ismail AA (2002). Maximum Likelihood Estimation and Optimal Design in Step Partially Accelerated Life Tests for the Pareto Distribution with Type-I Censoring. Proceedings of the 14th annual conference on Statistics and Computer Modeling in Human and Social Sciences, Faculty of Economics and Political Science, Cairo, University, Egypt, pp. 16 - 29.

Abdel-Ghani MM (2004). The Estimation Problem of the Log-Logistic Parameters in Step Partially Accelerated Life Tests using Type-I Censored Data. National Rev. Soc. Sci., 41(2): 1-19.

Aly HM, Ismail AA (2008). Optimum simple time-step stress plans for partially accelerated life testing with censoring. Far East J. Theor. Stat. 24(2): 175-200.

Ananda MM, Dalpatadu RJ, Singh AK (1996). Adaptive Bayes Estimators for Parameters of the Gompertz Survival Model. Appl. Math. Computat., 75(2): 167-177.

Bai DS, Chung SW (1992). Optimal Design of Partially Accelerated Life Tests for the Exponential Distribution under Type-I Censoring. IEEE Trans. on Reliability. 41(3): 400-406.

Bai DS, Chung SW, Chun YR (1993). Optimal Design of Partially Accelerated Life Tests for the Lognormal Distribution Under Type -I Censoring. Reliability Eng. syst. Safety., 40: 85-92.

Chen Z (1997). Parameter Estimation of the Gompertz Population. Biomet. J., 39: 117-124.

DeGroot MH, Goel PK (1979). Bayesian and Optimal Design in Partially Accelerated Life Testing. Naval research logistics quarterly. 16(2): 223-235.

Franses PH (1994). Fitting a Gompertz Curve. J. Operat. Res. Soc., 45: 109-113.

Garg ML, Rao BR, Redmond CK (1970). Maximum Likelihood Estimation of the Parameters of the Gompertz Survival Function. J. Royal Stat. Soc. C. 19: 152-159.

Goel PK (1971). Some Estimation Problems in the Study of Tamered Random Variables. Technical Report No. 50, Department of statistics. Carnegie-mellon University, Pittspurgh. Pennsylvania.

Gompertz B (1825). On the Nature of the Function Expressive of the Law of Human Mortality and on the New Mode of Determining the Value of Life Contingencies. Phil. Trans. R. Soc. A., 115: 513-580.

Gordon NH (1990). Maximum Likelihood Estimation for Mixutres of two Gompertz Distributions when Censoring Occurs. Communication in Statistics – Simulation & Computation, 19: 733-747.

Grimshaw SW (1993). Computing Maximum Likelihood Estimates for the Generalized Pareto Distribution. Technometrics, 35: 185-191.

Ismail AA (2010). Bayes estimation of Gompertz distribution parameters and acceleration factor under partially accelerated life tests with type-I censoring. J. Stat. Computat. Simulat., 80(11): 1253-1264.

Jaheen ZF (2003). Bayesian Prediction under a Mixture of Two-Component Gompertz Lifetime Model. Sociedad Espanola de Estadistica e Investigacion Operativa Test, 12(2): 413-426.

Johnson NL, Kotz S, Balakrishnan N (1994). Continuous Univariate Distributions 1. 2nd ed. John Wiley and Sons, pp. 25-26, 81-85.
Johnson NL, Kotz S, Balakrishnan N (1995). Continuous Univariate Distributions 2. 2nd ed. John Wiley and Sons.
Madi MT (1997). Bayesian Inference for Partially Accelerated Life Tests using Gibbs Sampling. Microelectronics and Reliability, 37(8): 1165-1168.
Makany R (1991). A Theoretical Basis of Gompertz's Curve. Biomet. J., 33: 121-128.
Nelson W. (1990). Accelerated Life Testing: Statistical Models, Data Analysis and Test Plans. John Wiley and sons, New York.
Osman MI (1987). A New Model for Analyzing the Survival of Heterogeneous Data. Ph.D. Thesis. Case Western Reserve University, USA.
Rao BR, Damaraju CV (1992). New Better than Used and other Concepts for a Class of Life Distribution. Biomet. J., 34: 919-935.
Read CB (1983). Gompertz Distribution. Encyclopedia of Statistical Sciences. Wiley, New York.
Tahir M (2003). Estimation of the Failure Rate in a Partially Accelerated Life Test: A Sequential Approach. Stochast. Anal. Appl., 21(4): 909-915.
Walker SG, Adham SA (2001). A Multivariate Gompertz-Type Distribution. J. Appl. Stat., 28(8): 1051-1065.
Wu JW, Hung WL, Tsai CH (2003). Estimation of the Parameters of the Gompertz Distribution under the First Failure-Censored Sampling Plan. Statistics, 37(6): 517-525.
Wu JW, Lee WC (1999). Characterization of the Mixtures of Gompertz Distributions by Conditional Expectation of Order Statistics. Biomet. J., 41: 371-381.
Xiong C, Ji M (2004). Analysis of Grouped and Censored Data From Step-Stress Life Test. IEEE Trans. Reliability., 53(1): 22-28.