academicJournals

Vol. 8(20), pp. 791-794, 25 May, 2013
DOI 10.5897/SRE09.294
ISSN 1992-2248 © 2013 Academic Journals
http://www.academicjournals.org/SRE

Scientific Research and Essays

Full Length Research Paper

Moments of the number of triangular configurations in binomial trials

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Accepted 29 November, 2010

Consider a matrix of locations in rows and columns, and suppose that at each location a binomial trial materializes independently which results in a specified event E with some probability. One possible configuration is that the event occurs at three adjacent locations, which we call a triangular configuration. The number of these triangular configurations is a random variable with a highly complex distribution. This paper determines the factorial moments of this distribution by the application of Memon's theorem.

Key words: Binomial trial, factorial moments.

INTRODUCTION

Single binomial trials often find their application in scientific inquiries especially where a trial results in some specified event with a constant probability and for any reason a researcher concentrates on the number of such events that happen randomly when the trial is independently repeated. Suppose that we have a set of locations arranged in m rows and n columns, and that binomial trials occur simultaneously at all these locations. If each trial results in a specified event, E with some probability, various configurations are likely to emerge. One could possibly be concerned about the probability of a particular configuration of these events.

Moran (1948), Fuchs and David (1965), Memon and David (1968) and Iqbal and Memon (2009) studied the distribution of various patterns that arose in similar situation based on binomial trials. This paper focuses on a configuration that the event occurs at three adjacent locations of an $n \times n$ lattice. It is called a triangular configuration, or a triangle, and introduces a random variable indicating the number of these triangles that materialize when the trials simultaneously occur. The distribution of this above random variable is investigated by applying Memon and David (1968) factorial moments theorem.

MODELING THE TRIANGLE

Links between positions and different possible triangles are defined as follows:

a) Let $\phi_{ij,i(j+1)}$ denote j^{th} link in i^{th} row with a value:
= 1 (if the event E occurs at locations j and j+1 in i^{th} row)
= 0 Otherwise

Let $\psi_{ij,(i+1)j}$ denote i^{th} link in j^{th} column with a value :
= 1 (if the event E occurs at locations i and i+1 in j^{th} column)
= 0 Otherwise

Let $\xi_{i(j+1),(i+1)j}$ denote diagonal link appearing after joining the locations (i+1,j) and

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(i,j+1) with a value:
= 1 (if the event E occurs at locations (i+1, j) and (i, j+1) locations)
= 0 Otherwise

So that $\phi_{ij,i(j+1)} = \psi_{ij,(i+1)j} = \xi_{i(j+1),(i+1)j} = 1$ for a triangle with the event E

Where i = 1, 2, 3, . . . , n – 1
j = 1, 2, 3, . . . , n – 1

Occurring at the locations (i, j), (i, j + 1), (i + 1, j)

b) Let $\phi_{(i+1)j,(i+1)(j+1)}$ denote j^{th} link in $(i+1)^{th}$ row with a value :
= 1 (if the event E occurs at locations j and j+1 in $(i+1)^{th}$ row)
= 0 Otherwise

Let $\psi_{i(j+1),(i+1)(j+1)}$ denote i^{th} link in j^{th} column with a value :
= 1 (if the event E occurs at locations i and i+1 in $(j+1)^{th}$ column)
= 0 Otherwise.

Let $\xi_{i(j+1),(i+1)j}$ denote diagonal link appearing after joining the locations (i+1,j) and (i,j+1) with a value:
= 1 (if the event E occurs at locations (i+1, j) and (i, j+1) locations)
= 0 Otherwise
So that $\phi_{(i+1)j,(i+1)(j+1)} = \psi_{i(j+1),(i+1)(j+1)} = \xi_{i(j+1),(i+1)j} = 1$ for a triangle with the event E

Where i = 1, 2, 3, . . . , n – 1
j = 1, 2, 3, . . . , n – 1

Occurring at the locations (i, j), (i, j+1), (i + 1, j+1).

c) Let $\phi_{(i+1)j,(i+1)(j+1)}$ denote j^{th} link in $(i+1)^{th}$ row with a value :
= 1 (if the event E occurs at locations j and j+1 in $(i+1)^{th}$ row)
= 0 Otherwise

Let $\psi_{ij,(i+1)j}$ denote i^{th} link in j^{th} column with a value :
= 1 (if the event E occurs at locations i and i+1 in j^{th} column)
= 0 Otherwise.

Let $\xi_{ij,(i+1)(j+1)}$ denote diagonal link appearing after joining the locations (i,j) and (i+1,j+1) with a value:
= 1 (if the event E occurs at locations (i, j) and (i+!, j+1) locations)
= 0 Otherwise.

So that $\phi_{(i+1)j,(i+1)(j+1)} = \psi_{ij,(i+1)j} = \xi_{ij,(i+1)(j+1)} = 1$ for a triangle with the event E

Where i = 1, 2, 3, . . . , n – 1
j = 1, 2, 3, . . . , n – 1

Occurring at the locations (i, j), (i+1, j), (i + 1, j+1)

d) Let $\phi_{ij,i(j+1)}$ denote j^{th} link in i^{th} row with a value :
= 1 (if the event E occurs at locations j and j+1 in i+1 row)
= 0 Otherwise

Let $\psi_{i(j+1),(i+1)(j+1)}$ denote i^{th} link in $(j+1)^{th}$ column with a value :
= 1 (if the event E occurs at locations i and i+1 in $(j+1)^{th}$ column)
= 0 Otherwise
Let $\xi_{ij,(i+1)(j+1)}$ denote diagonal link appearing after joining the locations (i,j) and (i+1,j+1) with a value:
= 1 (if the event E occurs at locations (i, j) and (i+!, j+1) locations)
= 0 Otherwise.

So that $\phi_{ij,i(j+1)} = \psi_{i(j+1),(i+1)(j+1)} = \xi_{ij,(i+1)(j+1)} = 1$ for a triangle with the event

Where i = 1, 2, 3, . . . , n – 1
j = 1, 2, 3, . . . , n – 1

E occurring at the locations (i, j), (i+1, j), (i + 1, j+1).

Factorial moment theorem

Memon and David theorem on factorial moments facilitate a relationship between factorial moments and probabilities of specified events. They consider n possibly dependent events each of whose materialization is determined by a single binomial trial. Then the r^{th} factorial moment of the number of materializing events is:

$$u_{[r]} = r! \sum_{{}^{n}c_{r}} P(w)$$

Where P(w) denotes the probability of materialization of all events in a set of size r and the summation extends over all $\binom{n}{r}$ sets of size r.

FACTORIAL MOMENTS OF THE RANDOM VARIABLE X

Let X be the random variable denoting the number of triangles that arise when independent binomial trials materialize at the same time at adjacent locations of an n x n lattice. Here the first three factorial moments are discussed.

First factorial moment

A particular triangle forms by the models a), b), c) and d).

For r = 1 in the above theorem the first factorial moment is given by:

$$u_{[1]} = \sum_k p_k$$

Where p_k is the probability of a particular triangle.

Since, here $p_k = p^3$

The first moment simplifies to:

$$4(n - 1)^2 p^3$$

Second factorial moment

For the second factorial moment

$$\mu_2 = 2! \sum_{{}^n c_r} \text{Prob (Two particular triangles appear at } n^2 \text{ locations)}$$

$$= a_{2,4}\, p^4 + a_{2,5}\, p^5 + a_{2,6}\, p^6$$

We find below the coefficients of probability p^4, p^5 and p^6.

Model for two triangles with a common side

1) $\phi_{ij,i(j+1)} = \phi_{(i+1)j,(i+1)(j+1)} = \psi_{ij,(i+1)j} = \xi_{i(j+1),(i+1)j} = 1$

Where i = 1, 2, 3, . . . , n – 1
j = 1, 2, 3, . . . , n – 1

The number of configurations is:

$n_1 = (n – 1)^2$

2) $\phi_{ij,i(j+1)} = \phi_{(i+1)j,(i+1)(j+1)} = \psi_{ij,(i+1)j} = \psi_{i(j+1),(i+1)(j+1)} = \xi_{ij,(i+1)(j+1)} = 1$

Where i = 1, 2, 3, . . . , n – 1
j = 1, 2, 3, . . . , n – 1

The number of configurations is:

$N_2 = (n - 1)^2$

3) $\phi_{(i+1)j,(i+1)(j+1)} = \psi_{i(j+1),(i+1)(j+1)} = \psi_{i(j+1),(i+2)(j+1)} = \xi_{(i+1)j,i(j+1)} = \xi_{(i+1)j,(i+2)(j+1)} = 1$

Where i = 1, 2, 3, . . . , n – 2
j = 1, 2, 3, . . . , n – 1

The number of configurations is:

$n_3 = 8(n – 1)(n - 2)$

$$a_{2,4} = 4(n-1)^2 + 16(n-1)(n-2)$$

Each pair occurs with probability p^4

Model for the two triangles with a common corner

4) $\phi_{ij,i(j+1)} = \phi_{i(j+1),i(j+2)} = \psi_{ij,(i+1)j} = \psi_{i(j+1),(i+1)(j+1)} = \xi_{(i+1)j,(i+1)(j+1)} = \xi_{(i+1)(j+1),i(j+2)} = 1$

Where i = 1, 2, 3, . . . , n – 1
j = 1, 2, 3, . . . , n – 2

The number of configurations is:

$n_4 = 16 (n – 1) (n - 2)$

5) $\phi_{ij,i(j+1)} = \phi_{(i+1)(j+1),(i+1)(j+2)} = \psi_{i(j+1),(i+1)(j+1)} = \psi_{(i+1)(j+2),(i+2)(j+2)} = \xi_{ij,(i+1)(j+1)} = \xi_{(i+1)(j+1),(i+2)(j+2)} = 1$

Where i = 1, 2, 3, . . . , n – 2
j = 1, 2, 3, . . . , n – 2

The number of configurations is :

$n_5 = 16(n – 2)^2$

$$a_{2,5} = 32(n-1)(n-2) + 32(n-2)^2$$

Each pair occurs with probability p^5

Model with probability p^6

The number of configurations is

$$a_{2,6} = 4(n - 1)^2 c_2 - \left(6(n-1)^2 + 24(n - 1)(n - 2) + 16(n-2)^2\right)$$

Third factorial moment

For the third factorial moment

$$\mu_{[3]} = 3! \sum_{{}^{n}c_{3}} \text{Prob (Three particular triangles appear at } n^{2} \text{ locations)}$$

$$= a_{3,5}\, p^{5} + a_{3,6}\, p^{6} + a_{3,7}\, p^{7} + a_{3,8}\, p^{8} + a_{3,9}\, p^{9}$$

We find the coefficients of p^5, p^6, p^7, p^8 and p^9.

Using the approach as for the second factorial moment we obtain

$$a_{3,5} = 16(n-1)(n-2)$$

$$a_{3,6} = 16(n-1)(n-2) + 16(n-2)(n-3)$$

$$a_{3,7} = 40(n-2)(n-3) + 48(n-1)(n-3) + 8(n-1)(n-2)^{2} + 16(n-1)(n-2)^{3}(n-3)$$

$$a_{3,8} = 80(n-2)^{2}(n-3) + 192(n-2)^{2}(n-3)^{3} + 96(n-2)(n-3)(n-4)^{3}$$

$$a_{3,9} = \binom{4(n-1)^{2}}{3} - \left(\begin{array}{l} 32(n-1)(n-2) + 56(n-2)(n-3) + \\ 48(n-1)(n-3) + 8(n-1)(n-2)^{2} + \\ 16(n-1)(n-2)^{3}(n-3) + 80(n-2)^{2}(n-3) + \\ 192(n-2)^{2}(n-3)^{3} + 96(n-2)(n-3)(n-4)^{3} \end{array} \right)$$

Fourth factorial moments

For the fourth factorial moment

$$\mu_{4} = 4! \sum_{{}^{n}c_{4}} \text{Prob (Four particular triangles appear at } n^{2} \text{ locations)}$$

$$= a_{4,6}\, p^{6} + a_{4,7}\, p^{7} + a_{4,8}\, p^{8} + a_{4,9}\, p^{9} + a_{4,10}\, p^{10} + a_{4,11}\, p^{11} + a_{4,12}\, p^{12}$$

we find the coefficients of p^6, p^7, p^8, p^9 p^{10}, p^{11} and p^{12}

Using the approach as for the second factorial moment we obtain

$$a_{4,6} = 32(n-1)(n-2)$$

$$a_{4,7} = 132(n-2)^{2} + 164(n-2)(n-4)$$

$$a_{4,8} = 132(n-1)(n-3) + 180(n-2)(n-3) + 192(n-1)(n-4) + 222(n-1)(n-3)^{2} + 144(n-1)(n-3)^{3}(n-4)$$

$$a_{4,9} = 192(n-1)(n-3)(n-4) + 252(n-4)^{2} + 90(n-2)^{2}(n-4) + 252(n-3)^{2}(n-4)^{3} + 192(n-2)(n-3)(n-4)^{3}$$

$$a_{4,10} = 132(n-3)^{2}(n-4) + 156(n-3)^{2}(n-4)^{2} + 120(n-2)^{2}(n-3)(n-4) + 196(n-3)^{2}(n-4)^{2}(n-5) + 168(n-2)(n-3)^{2}(n-4)^{3}$$

$$a_{4,11} = 162(n-3)^{2}(n-5)(n-6) + 132(n-3)^{2}(n-4)^{2}(n-5) + 126(n-4)^{2}(n-5)^{2}(n-5) + 72(n-3)^{3}(n-4)^{2}(n-5) + 48(n-4)^{2}(n-5)^{2}(n-6)^{3}$$

$$a_{4,12} = \binom{4(n-1)^{2}}{4} - \frac{1}{6}\left(a_{4,6} + a_{4,7} + a_{4,8} + a_{4,9} + a_{4,10} + a_{4,11}\right)$$

A REMARK ON ASYMPTOTIC MOMENTS

Assuming that $n^2p^3 \to \lambda$ as $n \to \infty$, it is not difficult to see that the first, second, third and fourth factorial moments of X are simplified to λ, λ^2, λ^3, λ^4, respectively indicating that X is asymptotically distributed as a Poisson random variable with λ as its parameter.

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